BOUNDS ON CONTINUOUS ENTANGLEMENT GAIN

SUBMITTED
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A final year project report
presented to
Nanyang Technological University
in partial fulfilment of the
requirements for the
Bachelor of Science (Hons) in Physics / Applied Physics
Nanyang Technological University

May 2018

"The first principle is that you must not fool yourself and you are the easiest person to fool."

Richard Feynman

"Strive not to be a success, but rather to be of value."

Albert Einstein

Acknowledgements

I would like to thank my FYP professor, Asst. Prof Paterek Tomasz, my FYP supervisor, Tanjung Krisnanda, and Ray Fellix Garnardi for all the help and guidance they have given me. They have been very patient and careful in correcting many misunderstandings I had along the way. This thesis would not be possible without them.

I would also like to thank my friends and family for the support they have given me throughout this FYP. I would like to especially mention my gratitude towards Samuel Pang, Shelvia Wongso, Lim Hui Wen, Darieca Lim, Jonathan Lau, Lim Leyu and Atikah, for listening to my woes and frustrations.

Abstract

Entanglement is a physical resource that is important in quantum teleportation, quantum dense coding and quantum cryptography. In this thesis, we investigated entanglement distribution between particles A and B (possibly located in different laboratories) via continuous interaction with an ancilla, C. We assume that A and B do not interact directly with each other, but only via C, and therefore the total Hamiltonian is of the form $H_{AC} + H_{BC}$. Our first result is the simplification of the expressions for H_{AC} and H_{BC} for a class of commuting Hamiltonians, i.e. $[H_{AC}, H_{BC}] = 0$ in which H_{AC} is neither a free Hamiltonian on A nor a free Hamiltonian on C (which implies that A and C interact), and likewise H_{BC} is neither a free Hamiltonian on B nor a free Hamiltonian on C.

Using these simplifications, we looked at the time evolution of pure product states $|\alpha\beta\gamma\rangle$ and bi-product states $|\chi\rangle_{AB} |\gamma\rangle_C$. We were able to analytically prove for pure product states that entanglement $A : BC$ (or $B : AC$) is bounded by entanglement $AB : C$, that is the amount of entanglement in C. For bi-product states, we found a promising bound stating that entanglement gain, i.e. entanglement at time t minus initial entanglement is bounded by the entanglement in C. This is confirmed by extensive numerical simulations.

We also considered the case where H_{AC} realizes the swap operator S_{A-C} at a particular time and swaps the state of A with C while H_{BC} is just the identity operator. This scenario falls outside the class considered above. For this case, we managed to prove analytically (for bi-product states $|\chi\rangle_{AB} |\gamma\rangle_C$) the same bound that we conjectured above.

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Chapter 1

Introduction

1.1 Motivations and objectives

Entanglement, most famously described by Einstein as 'spooky action at a distance', is realized to be a resource as physical as energy [1]. Entanglement unlocks new range of possibilities such as quantum teleportation [2], quantum dense coding [3] and quantum cryptography [4]. Since the discovery of entanglement by Einstein *et al.* [5] and Schrödinger [6], much research has been carried out in the last few decades to understand the rules at play that govern entanglement.

There are various ways for entanglement to be created between two systems, A and B. This thesis focuses on a particular method where A and B are allowed to interact continuously with an ancilla, C, but do not directly interact with each other (see Figure 1.1).

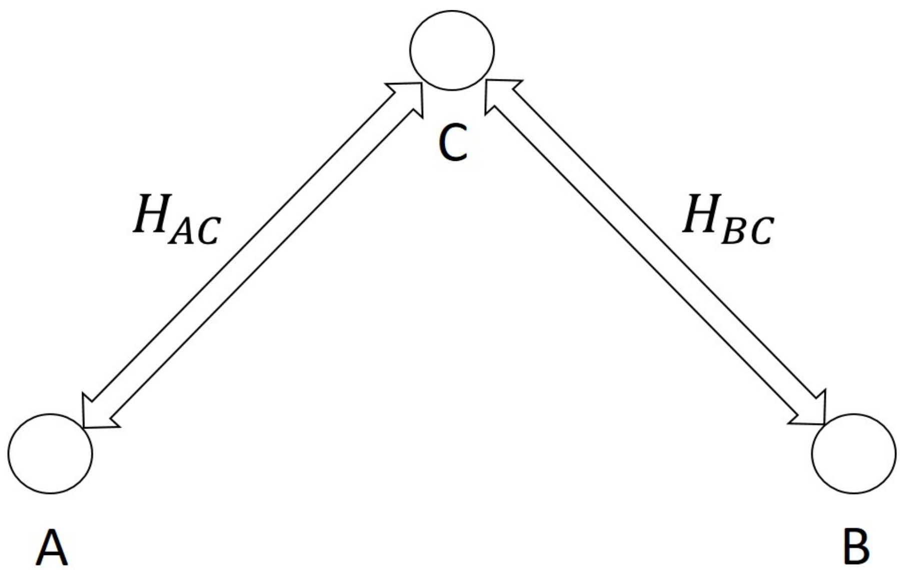

Figure 1.1: Setup for distributing entanglement between A and B via continuous interaction with C.

Surprisingly, Cubitt *et al.* [7] showed that it is possible to distribute entanglement to A and B without C ever being entangled. In other words, it is possible to distribute entanglement without actually communicating entanglement. This goes against our intuition that in order to distribute entanglement to A and B, C must first carry entanglement (i.e. C has to be entangled with AB). This phenomenon has been experimentally verified in [8–10]. However, the scenario in those experiments is different from the one in Figure 1.1. Namely, in those experiments there are no continuous interactions, rather particle C first interacts with A (via a quantum gate), then gets transmitted to another laboratory where it interacts with B.

The resources required to distribute entanglement in the setting of Figure 1.1 are at present mostly unknown and they are the main topic of this thesis. It is only known that in order to entangle A and B, there has to be quantum discord $D_{AB|C}$ at some time during the evolution [11]. Our overaching aim is to provide quantities which are upper bounding the amount of distributed entanglement.

1.2 Flow of the thesis

In this Section, we will give an overview of the flow of this thesis. Chapters 1 to 4 are a review of the standard quantities used in quantum information. They form the basis for understanding the terminology and formulas that we will use in this thesis.

In Chapter 2 we look at quantities related to quantum states, such as the density operator, purity, von Neumann entropy, how to rewrite quantum states in the Schmidt decomposition, and how to obtain the reduced density operators of subsystems via the partial trace. In Chapter 3 we look at the conditional entropy. We include the joint entropy theorem and the entropy of a tensor product because it enables us to rewrite the conditional entropy in a form that becomes useful to us in our proof. Strong subadditivity and mutual information are also reviewed since they are an integral part of our proofs. In Chapter 4 review what it means for pure states and mixed states to be separable (i.e. not entangled). Following that, we introduce the relative entropy of entanglement, which is an entanglement measure that we will be using in this thesis.

Chapter 5 starts by introducing the setup for entanglement distribution via continuous interaction with an ancilla. We ask what it means for Hamiltonians H_{AC} and H_{BC} to commute. Theorem 1 presents the answer to that question. It turns out that if $[H_{AC}, H_{BC}] = 0$ there is a common eigenbasis which is *product* on C, and we also show that H_{AC} and H_{BC} can be re-expressed as $H_{AC} = H_A + \tilde{H}_A \otimes \tilde{H}_C$ and $H_{BC} = H_B + \tilde{H}_B \otimes \tilde{H}_C$. This simplification does not hold for all Hamiltonians $H_{AC} + H_{BC}$ but for a particular subclass that we clarify. For simplicity, we shall call this subclass of Hamiltonians $\mathcal{H}_1$.

Next, we introduce a particular case where H_{AC} generates the swap operator, S_{A-C}, at some particular time. We will take H_{BC} to be the identity operator, so H_{AC} and H_{BC} commute as well. For simplicity, we shall call this pair of Hamiltonians $\mathcal{H}_2$. It turns out that for this trivial case where B is effectively not interacting with C, there are some derivable bounds on entanglement between $A : BC$ and $AB : C$ that are pretty suggestive.

For the subclass of Hamiltonians $\mathcal{H}_1$, we show in Theorem 2 (for initial pure product states) that entanglement $A : BC$ and $B : AC$ is respectively bounded by the amount of entanglement in the ancilla, C. Next, we consider bi-product states where A and B are initially entangled ($|\chi\rangle_{AB} |\gamma\rangle_C$). We find numerically a promising bound after randoming a total of 1 million samples,

$$\begin{aligned} S_A(t) - S_A(0) &\leq S_C(t) \\ S_B(t) - S_B(0) &\leq S_C(t) \end{aligned} \tag{1.2.1}$$

where S_A refers to the von Neumann entropy of A. While the bound is very intuitive, we have not been able to prove it analytically. Despite that, we showed that $S_A(t) \geq S_A(0)$ and $S_B(t) \geq S_B(0)$.

For the pair of Hamiltonians $\mathcal{H}_2$, B is effectively not interacting with C, which goes against our initial intention that in our setup, A and B are continuously interacting with C. Yet, we managed to derive $|S_A(t) - S_A(0)| \leq S_C(t)$ for initial bi-product states $|\chi\rangle_{AB} |\gamma\rangle_C$. This is a suggestive result that supports what we have not been able to prove so far for the subclass of Hamiltonians $\mathcal{H}_1$. We also applied the Schmidt decomposition on operators to the case where $\tilde{H}_{AC}$ is the interaction part of the Hamiltonian that generates S_{A-C} at a particular time. We showed that it cannot be written as a single product (i.e. $\tilde{H}_{AC} \neq \tilde{H}_A \otimes \tilde{H}_C$).

Chapter 2

Quantum states

This Chapter will review some definitions and concepts related to quantum states [12]. Section 2.1 introduces pure and mixed states, Section 2.2 reviews the von Neumann entropy, Section 2.3 reviews the Schmidt decomposition for bipartite pure states and extends the definition to tripartite pure states, and Section 2.4 reviews the partial trace.

2.1 Pure and mixed states

A quantum system whose state is known exactly is said to be in a pure state $|\psi\rangle$. In this case the density operator is simply the outer product of the state with itself, $\rho = |\psi\rangle\langle\psi|$. Otherwise a state is called a mixed state. Mixed states are mixtures of different pure states, $|\psi_i\rangle$, and the density operator reads $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i| = \sum_i \lambda_i \rho_i$, where λ_i are probabilities and ρ_i are the density operators of those different pure states. While $\{|\psi_i\rangle\}$ may not be orthonormal to each other, one can always write ρ in diagonal representation, i.e. $\rho = \sum_j \mu_j |\phi_j\rangle\langle\phi_j|$, where $\{|\phi_j\rangle\}$ form an orthonormal basis and μ_j are the new probabilities that fall into the diagonal representation. To determine whether a given ρ is pure or mixed, we calculate $\mathrm{tr}(\rho^2)$.

If ρ is pure,

$$
\begin{aligned}
\mathrm{tr}(\rho^2) &= \mathrm{tr}(|\psi\rangle\langle\psi|\psi\rangle\langle\psi|) \\
&\quad | \quad \text{Using the cyclic property of the trace, } \mathrm{tr}(|\psi\rangle\langle\phi|) = \langle\phi|\psi\rangle\,. \\
&= \langle\psi|\psi\rangle\langle\psi|\psi\rangle \\
&\quad | \quad \text{By the normalization condition, } \langle\psi|\psi\rangle = 1. \\
&= 1. \qquad (2.1.1)
\end{aligned}
$$

If ρ is mixed,

$$\begin{aligned}
\operatorname{tr}(\rho^2) &= \operatorname{tr}(\sum_j \mu_j \left|\phi_j\right\rangle \left\langle\phi_j\right| \sum_k \mu_k \left|\phi_k\right\rangle \left\langle\phi_k\right|) \\
&= \operatorname{tr}(\sum_{jk} \mu_j \mu_k \left|\phi_j\right\rangle \left\langle\phi_j|\phi_k\right\rangle \left\langle\phi_k\right|) \\
&\mid \quad \text{Using the linearity of the trace.} \\
&= \sum_{jk} \mu_j \mu_k \operatorname{tr}(\left|\phi_j\right\rangle \left\langle\phi_j|\phi_k\right\rangle \left\langle\phi_k\right|) \\
&\mid \quad \text{Using the cyclic property of the trace, } \operatorname{tr}(\left|\psi\right\rangle \left\langle\phi\right|) = \left\langle\phi|\psi\right\rangle. \\
&= \sum_{jk} \mu_j \mu_k |\left\langle\phi_k|\phi_j\right\rangle|^2 \\
&\mid \quad \text{For } j = k,\ |\left\langle\phi_k|\phi_j\right\rangle|^2 = 1. \text{ For } j \neq k,\ |\left\langle\phi_k|\phi_j\right\rangle|^2 = 0. \\
&= \sum_j \mu_j^2 \\
&< 1. \qquad (2.1.2)
\end{aligned}$$

2.2 The von Neumann entropy

The von Neumann entropy of a quantum state ρ is defined as

$$S(\rho) \equiv -\text{tr}(\rho\text{log}\rho), \tag{2.2.1}$$

where the logarithms are taken to base two. If λ_i are the eigenvalues of ρ and $|e_i\rangle$ are the corresponding orthonormal eigenvectors, then the von Neumann entropy has a more convenient expression:

$$\begin{aligned}
S(\rho) &= -\text{tr}(\sum_i \lambda_i |e_i\rangle \langle e_i| \log(\sum_j \lambda_j |e_j\rangle \langle e_j|)) \\
&\quad | \ \text{Expressing } \rho \text{ in eigendecomposition.} \\
&= -\text{tr}(\sum_i \lambda_i |e_i\rangle \langle e_i| \sum_j \text{log}\lambda_j |e_j\rangle \langle e_j|) \\
&\quad | \ \text{The logarithm of a density matrix is equivalent to the logarithm of its eigenvalues.} \\
&= -\text{tr}(\sum_{ij} \lambda_i\text{log}\lambda_j |e_i\rangle \langle e_i|e_j\rangle \langle e_j|) \\
&\quad | \ \text{Using the orthonormality of eigenvectors from eigendecompositon, } \langle e_i|e_j\rangle = \delta_{ij}. \\
&= -\text{tr}(\sum_i \lambda_i\text{log}\lambda_i |e_i\rangle \langle e_i|) \\
&\quad | \ \text{Using the linearity of the trace.} \\
&= -\sum_i \lambda_i\text{log}\lambda_i\text{tr}(|e_i\rangle \langle e_i|) \\
&\quad | \ \text{Using the cyclic property of the trace, tr}(|\psi\rangle \langle\phi|) = \langle\phi|\psi\rangle \text{, and } \langle e_i|e_i\rangle = 1. \\
&= -\sum_i \lambda_i\text{log}\lambda_i, \qquad (2.2.2)
\end{aligned}$$

where λ_i are probabilities.

The von Neumann entropy is zero (for pure states) or finite and positive (for mixed states). It is zero for pure states since pure states are entirely known. The density operator, ρ, is just $|\psi\rangle \langle\psi|$, which is already the diagonal representation with a probability of one. For mixed states, there are at least two non-zero λ_i, hence the von Neumann entropy is positive.

2.3 Schmidt decomposition

If the state of a bipartite system is pure, Schmidt decomposition guarantees that we can write

$$|\psi\rangle_{AB} = \sqrt{\lambda}\,|a\rangle_A\,|b\rangle_B + \sqrt{1-\lambda}\,|a_\perp\rangle_A\,|b_\perp\rangle_B \tag{2.3.1}$$

where λ are probabilities, $\{|a\rangle, |a_\perp\rangle\}$ are orthonormal states on A, and $\{|b\rangle, |b_\perp\rangle\}$ are orthonormal states on B.

If we have a tripartite system and the quantum state describing this system is $|\psi\rangle_{ABC}$, we can simply group subsystems A and B together into a single subsystem, leaving subsystem C alone. The Schmidt decomposition allows us to write,

$$|\psi\rangle_{ABC} = |\psi\rangle_{(AB)C} = \sqrt{\lambda_i}\,|\chi\rangle_{AB}\,|c\rangle_C + \sqrt{1-\lambda_i}\,|\chi_\perp\rangle_{AB}\,|c_\perp\rangle_C \tag{2.3.2}$$

where λ_i are probabilities, $\{|\chi\rangle, |\chi_\perp\rangle\}$ are orthonormal states on AB, and $\{|c\rangle, |c_\perp\rangle\}$ are orthonormal states on C.

Taking the partial trace (skip ahead to Section 2.4 for the partial trace), we find

$$\begin{aligned}\rho_{AB} &= \lambda_i\,|\chi\rangle\langle\chi| + (1-\lambda_i)\,|\chi_\perp\rangle\langle\chi_\perp|\,,\\ \rho_C &= \lambda_i\,|c\rangle\langle c| + (1-\lambda_i)\,|c_\perp\rangle\langle c_\perp|\,.\end{aligned} \tag{2.3.3}$$

Since ρ_{AB} and ρ_C are already in eigendecomposition, we can easily find that their von Neumann entropies are the same.

$$S(\rho_{AB}) = S(\rho_C) = -\lambda_i \log\lambda_i - (1-\lambda_i)\log(1-\lambda_i). \tag{2.3.4}$$

One also has the freedom to group subsystems B and C together, or A and C together. The Schmidt decomposition reads,

$$\begin{aligned}|\psi\rangle_{ABC} &= |\psi\rangle_{A(BC)} = \sqrt{\lambda_j}\,|a\rangle_A\,|\varphi\rangle_{BC} + \sqrt{1-\lambda_j}\,|a_\perp\rangle_A\,|\varphi_\perp\rangle_{BC}\,,\\ |\psi\rangle_{ABC} &= |\psi\rangle_{B(AC)} = \sqrt{\lambda_k}\,|b\rangle_B\,|\Phi\rangle_{AC} + \sqrt{1-\lambda_k}\,|b_\perp\rangle_B\,|\Phi_\perp\rangle_{AC}\,,\end{aligned} \tag{2.3.5}$$

where $\lambda_i \neq \lambda_j \neq \lambda_k$.

Similarly, one can calculate the von Neumann entropies of the corresponding subsystems and find them to be equal.

$$\begin{aligned} S(\rho_A) &= S(\rho_{BC}) = -\lambda_j \log\lambda_j - (1-\lambda_j)\log(1-\lambda_j), \\ S(\rho_B) &= S(\rho_{AC}) = -\lambda_k \log\lambda_k - (1-\lambda_k)\log(1-\lambda_k). \end{aligned} \tag{2.3.6}$$

Since all pure states can be written in Schmidt decomposition, the above holds for all pure states.

2.4 Partial trace

The partial trace is the operation that takes us from the density operator of a composite system to the density operator describing a particular subsystem. Let us consider a pure state $|\psi\rangle$ of a bipartite system, AB. From Section 2.3, the Schmidt decomposition allows us to write,

$$|\psi\rangle = \sqrt{\lambda}\, |a\rangle_A |b\rangle_B + \sqrt{1-\lambda}\, |a_\perp\rangle_A |b_\perp\rangle_B . \tag{2.4.1}$$

To find the reduced density operator for subsystem A, we take the partial trace with respect to subsystem B. It is written as

$$\begin{aligned} \rho_A &= \mathrm{tr}_B(|\psi\rangle\langle\psi|) \\ &= \mathrm{tr}_B(\lambda\, |a\rangle_A \langle a| \otimes |b\rangle_B \langle b| + \sqrt{\lambda(1-\lambda)}(|a\rangle_A \langle a_\perp| \otimes |b\rangle_B \langle b_\perp| + |a_\perp\rangle_A \langle a| \otimes |b_\perp\rangle_B \langle b|)\ + \\ &\quad (1-\lambda)\, |a_\perp\rangle_A \langle a_\perp| \otimes |b_\perp\rangle_B \langle b_\perp|) \\ &\mid \ \text{where the trace only acts on } B \text{ and we use the orthonormality between } |b\rangle \text{ and } |b_\perp\rangle. \\ &= \lambda\, |a\rangle\langle a| + (1-\lambda)\, |a_\perp\rangle\langle a_\perp| . \end{aligned} \tag{2.4.2}$$

Likewise, the reduced density operator for subsystem B reads

$$\begin{aligned} \rho_B &= \mathrm{tr}_A(|\psi\rangle\langle\psi|) \\ &= \mathrm{tr}_A(\lambda\, |a\rangle_A \langle a| \otimes |b\rangle_B \langle b| + \sqrt{\lambda(1-\lambda)}(|a\rangle_A \langle a_\perp| \otimes |b\rangle_B \langle b_\perp| + |a_\perp\rangle_A \langle a| \otimes |b_\perp\rangle_B \langle b|)\ + \\ &\quad (1-\lambda)\, |a_\perp\rangle_A \langle a_\perp| \otimes |b_\perp\rangle_B \langle b_\perp|) \\ &\mid \ \text{where the trace only acts on } A \text{ and we use the orthonormality between } |a\rangle \text{ and } |a_\perp\rangle. \\ &= \lambda\, |b\rangle\langle b| + (1-\lambda)\, |b_\perp\rangle\langle b_\perp| . \end{aligned} \tag{2.4.3}$$

Similarly if the quantum state in a tripartite system is pure and we want to calculate the reduced density operator for subsystem AB and subsystem C, we can choose to apply the Schmidt decomposition in the following way, separating into AB and C partitions.

$$|\psi\rangle = \sqrt{\lambda_i}\, |\chi\rangle_{AB} \, |c\rangle_C + \sqrt{1-\lambda_i}\, |\chi_\perp\rangle_{AB} \, |c_\perp\rangle_C \,. \tag{2.4.4}$$

We find

$$\begin{aligned} \rho_{AB} &= \mathrm{tr}_C(|\psi\rangle \langle\psi|) \\ &= \lambda_i \, |\chi\rangle \langle\chi| + (1-\lambda_i) \, |\chi_\perp\rangle \langle\chi_\perp| \,, \qquad (2.4.5) \\ \rho_C &= \mathrm{tr}_{AB}(|\psi\rangle \langle\psi|) \\ &= \lambda_i \, |c\rangle \langle c| + (1-\lambda_i) \, |c_\perp\rangle \langle c_\perp| \,, \qquad (2.4.6) \end{aligned}$$

analogous to what we had before.

Chapter 3

Entropies

This Chapter reviews entropic quantities such as the conditional entropy, mutual information and strong subadditivity. From [12], we show the proofs for the joint entropy theorem and the entropy of a tensor product. This derivation allows us to rewrite the conditional entropy in a form that becomes useful to us in Chapter 5.

3.1 Conditional entropy

Classically, the conditional entropy $H_{X|Y}$, is a measure of how uncertain we are on average about X given that we have complete knowledge about Y. It is denoted as

$$H_{X|Y} = H_{XY} - H_Y, \tag{3.1.1}$$

where H_{XY} is the Shannon entropy of the composite system XY and H_Y is the Shannon entropy of Y. It can be visualized in Fig 3.1. It is zero if X and Y are perfectly correlated, and one if X and Y are statistically independent. Classically, the conditional entropy is always non-negative, however quantum mechanically, this lower bound can be violated. The quantum conditional entropy $S_{A|B}$ is defined as

$$S_{A|B} = S_{AB} - S_B. \tag{3.1.2}$$

Consider Bell states. The composite system AB is pure so $S_{AB} = 0$, whereas individual subsystems are maximally mixed so $S_A = S_B = 1$. Therefore, quantum mechanically, it is possible for $S_{A|B}$ to be negative.

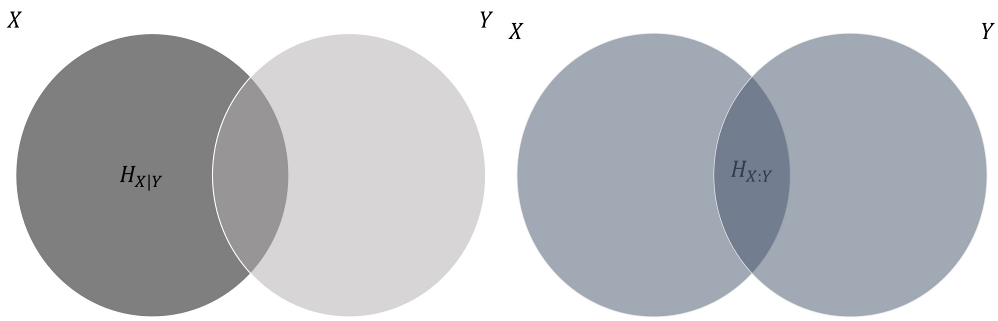

Figure 3.1: Visualization of the classical conditional entropy and mutual information.

3.1.1 Joint entropy theorem

Suppose ρ is the density operator for a composite system AB, such that

$$\rho = \sum_i \sigma_i \otimes p_i |i\rangle \langle i| = \sum_i p_i \sigma_i \otimes |i\rangle \langle i|, \tag{3.1.3}$$

where σ_i is any set of density operators on subsystem A, p_i are probabilities, and $|i\rangle$ are orthonormal states on subsystem B. Let $\lambda_j^{(i)}$ and $|e_j\rangle^{(i)}$ be the eigenvalues and corresponding eigenvectors of σ_i such that the eigendecomposition of σ_i is

$$\sigma_i = \sum_j \lambda_j^{(i)} |e_j\rangle^{(i)} \langle e_j|^{(i)}. \tag{3.1.4}$$

Then the joint entropy theorem states that,

$$\begin{aligned} S(\rho) &= S\left(\sum_i p_i \sum_j \lambda_j^{(i)} |e_j\rangle^{(i)} \langle e_j|^{(i)} \otimes |i\rangle \langle i|\right) \\ &= S\left(\sum_{ij} p_i \lambda_j^{(i)} |e_j\rangle^{(i)} |i\rangle \langle e_j|^{(i)} \langle i|\right) \\ &\quad | \quad \text{where } p_i\lambda_j^{(i)} \text{ are the new eigenvalues and } |e_j\rangle^{(i)} |i\rangle \text{ are the new eigenvectors} \\ &= -\sum_{ij} p_i \lambda_j^{(i)} \log p_i \lambda_j^{(i)} \\ &= -\sum_{ij} p_i \lambda_j^{(i)} (\log p_i + \log \lambda_j^{(i)}) \\ &= -\sum_i p_i \log p_i \sum_j \lambda_j^{(i)} - \sum_i p_i \sum_j \lambda_j^{(i)} \log \lambda_j^{(i)} \\ &\quad | \quad \text{where } \sum_j \lambda_j^{(i)} = 1 \text{ and } -\sum_j \lambda_j^{(i)} \log \lambda_j^{(i)} = S(\sigma_i) \\ &= -\sum_i p_i \log p_i + \sum_i p_i S(\sigma_i). \end{aligned} \tag{3.1.5}$$

Now suppose ρ is a density operator of a composite system ABC, such that

$$\rho = \sum_i \sigma_i^A \otimes \sigma_i^B \otimes p_i \left|i\right\rangle_C \left\langle i\right|, \tag{3.1.6}$$

then $S_{AB|C}$ reads,

$$\begin{aligned} S_{AB|C} &= S_{ABC} - S_C \\ &= S\left(\sum_i (\sigma_i^A \otimes \sigma_i^B) \otimes p_i \left|i\right\rangle_C \left\langle i\right|\right) - S\left(\sum_i p_i \left|i\right\rangle_C \left\langle i\right|\right) \\ &| \text{ Using the joint entropy theorem for the first term.} \\ &= -\sum_i p_i \log p_i + \sum_i p_i S(\sigma_i^A \otimes \sigma_i^B) - (-\sum_i p_i \log p_i) \\ &= \sum_i p_i S(\sigma_i^A \otimes \sigma_i^B). \end{aligned} \tag{3.1.7}$$

3.1.2 Entropy of a tensor product

It turns out that Eq. (3.1.7) can be simplified further. Consider a joint system AB where both subsystems are in product states, then it is natural to think that the entropy of joint system AB is just the sum of the entropies of the respective subsystems. The proof goes as follows:

Let ρ be the density operator for subsystem A and σ be the density operator for subsystem B. Then the joint system AB will be represented by the tensor product, $\rho \otimes \sigma$.

Let λ_i and $|\mu_i\rangle$ be the eigenvalues and corresponding eigenvectors of ρ such that the eigendecomposition of ρ is

$$\rho_A = \sum_i \lambda_i \left|\mu_i\right\rangle \left\langle \mu_i\right|. \tag{3.1.8}$$

Likewise, let λ_j and $|\nu_j\rangle$ be the eigenvalues and corresponding eigenvectors of σ such that the eigenvalue decomposition of σ is

$$\sigma = \sum_j \lambda_j \left|\nu_j\right\rangle \left\langle \nu_j\right|. \tag{3.1.9}$$

Then the joint system AB is just:

$$\begin{aligned} \rho \otimes \sigma &= \sum_{ij} \lambda_i \lambda_j |\mu_i\rangle \left\langle \mu_i\right| \otimes \left|\nu_j\right\rangle \left\langle \nu_j\right| \\ &= \sum_{ij} \lambda_i \lambda_j \left|\mu_i \nu_j\right\rangle \left\langle \mu_i \nu_j\right|, \end{aligned} \tag{3.1.10}$$

where $\lambda_i\lambda_j$ and $|\mu_i\nu_j\rangle$ are the eigenvalues and corresponding eigenvectors of $\rho \otimes \sigma$. It follows that the entropy of the joint system AB is:

$$\begin{aligned}
S(\rho \otimes \sigma) &= -\sum_{ij} \lambda_i\lambda_j \text{log}\lambda_i\lambda_j \\
&= -\sum_{ij} \lambda_i\lambda_j(\text{log}\lambda_i + \text{log}\lambda_j) \\
&= -\sum_j \lambda_j \sum_i \lambda_i \text{log}\lambda_i - \sum_i \lambda_i \sum_j \lambda_j \text{log}\lambda_j \\
&\mid \quad \text{Using } \sum_i \lambda_i = \sum_j \lambda_j = 1. \\
&= S(\rho) + S(\sigma).
\end{aligned} \tag{3.1.11}$$

With this, we can rewrite Eq.(3.1.7) as,

$$\begin{aligned}
S_{AB|C} &= \sum_i p_i S(\sigma_i^A \otimes \sigma_i^B) \\
&= \sum_i p_i (S(\sigma_i^A) + S(\sigma_i^B)),
\end{aligned} \tag{3.1.12}$$

which will be useful in Lemma 2.

3.2 Strong subadditivity

Strong subadditivity is proven in [12] and it reads

$$S_{ijk} + S_j \leq S_{ij} + S_{jk}, \tag{3.2.1}$$

where $\{i, j, k\}$ is any permutation of $\{A, B, C\}$.

3.3 Mutual information

Classically, the mutual information, $H_{X:Y}$, measures the amount of information shared between X and Y. It is reads,

$$H_{X:Y} = H_X + H_Y - H_{XY}. \tag{3.3.1}$$

The mutual information is zero if X and Y are statistically independent, and one if X and Y are perfectly correlated. It is always non-negative by defintion. Figure 3.1 shows a visual illustration of the mutual information. Classically, the amount of information in a composite system can never be more than the amount of information carried by its subsystems (i.e. $0 \leq H_{X:Y} \leq 1$). However, quantum mechanically, this is not the case. The quantum mutual information reads,

$$S_{X:Y} = S_X + S_Y - S_{XY}. \tag{3.3.2}$$

Consider Bell states. The composite system AB is pure ($S_{AB} = 0$), whereas individual subsystems are maximally mixed ($S_A = S_B = 1$). This means that quantum mechanically, it is possible to have complete knowledge about the composite system and have complete uncertainty about its subsystems ($S_{X:Y} = 2$ for Bell states).

Chapter 4

Separability and the relative entropy of entanglement

This Chapter reviews what separability means in the case of bipartite states and extends this description to tripartite states. We also introduce the relative entropy of entanglement, which is the entanglement measure that we are using to quantify entanglement in this thesis.

4.1 Separability for pure states

Suppose the quantum state of a composite system AB is pure. Separability just means that

$$|\psi\rangle_{AB} = |\alpha\rangle_A \otimes |\beta\rangle_B \,. \tag{4.1.1}$$

This means that A has a definite state, $|\alpha\rangle$, and B has a definite state, $|\beta\rangle$.

In tripartite systems, separability can come in various forms. A quantum state can be bi-separable or completely separable. Suppose the quantum state of a composite system ABC is pure. If it is completely separable, it can be written as

$$|\psi\rangle_{ABC} = |\alpha\rangle_A \otimes |\beta\rangle_B \otimes |\gamma\rangle_C \,. \tag{4.1.2}$$

This just means that A, B and C each have a definite state and that they behave locally as three completely independent subsystems.

If the tripartite pure state is bi-separable, one must indicate in which partitions the state is separable in. For example, a state that is AB:C separable reads

$$|\psi\rangle_{ABC} = |\chi\rangle_{AB} \otimes |\gamma\rangle_{C} \,. \tag{4.1.3}$$

The state of subsystem AB and the state of subsystem C are pure and independent, however entanglement can be present between subsystems A and B.

Likewise,

$$\begin{aligned} &\text{A:BC separable:} \;\; |\psi\rangle_{ABC} = |\alpha\rangle_{A} \otimes |\chi\rangle_{BC} \,, \\ &\text{AC:B separable:} \;\; |\psi\rangle_{ABC} = |\chi\rangle_{AC} \otimes |\beta\rangle_{B} \,. \end{aligned} \tag{4.1.4}$$

4.2 Separability for mixed states

For bipartite mixed states, the condition for separability is analogous to that of bipartite pure states. Recall that a mixed state is a mixture of different pure states, written as

$$\rho = \sum_i \lambda_i |\psi_i\rangle \langle\psi_i| \,. \tag{4.2.1}$$

A mixed state ρ_{AB} is said to be separable if there exists a decomposition where all its different pure states, $|\psi_i\rangle$, are separable. In other words, $|\psi_i\rangle_{AB} = |\alpha_i\rangle_A \otimes |\beta_i\rangle_B$.

$$\begin{aligned} \rho_{AB} &= \sum_i \lambda_i (|\alpha_i\rangle_A \otimes |\beta_i\rangle_B)(\langle\alpha_i|_A \otimes \langle\beta_i|_B) \\ &= \sum_i \lambda_i (|\alpha_i\rangle_A \langle\alpha_i|_A) \otimes (|\beta_i\rangle_B \langle\beta_i|_B) \\ &= \sum_i \lambda_i \rho_A^{(i)} \otimes \rho_B^{(i)} \,. \end{aligned} \tag{4.2.2}$$

For tripartite mixed states, it is not entirely analogous to that for tripartite pure states. First, the part that is analogous is: If there exists a decomposition for ρ_{ABC} such that each of different pure states $|\psi_i\rangle$ are separable in the same partition, then ρ_{ABC} is said to be separable in that partition.

To illustrate,

$$
\begin{aligned}
\text{Completely separable: } \rho_{ABC} &= \sum_i \lambda_i (|\alpha_i\rangle_A \otimes |\beta_i\rangle_B \otimes |\gamma_i\rangle_C)(\langle\alpha_i|_A \otimes \langle\beta_i|_B \otimes \langle\gamma_i|_C) \\
&= \sum_i \lambda_i (|\alpha_i\rangle_A \langle\alpha_i|_A) \otimes (|\beta_i\rangle_B \langle\beta_i|_B) \otimes (|\gamma_i\rangle_C \langle\gamma_i|_C) \\
&= \sum_i \lambda_i \rho_A^{(i)} \otimes \rho_B^{(i)} \otimes \rho_C^{(i)}, \qquad (4.2.3) \\
AB{:}C \text{ separable: } \rho_{ABC} &= \sum_i \lambda_i (|\chi_i\rangle_{AB} \otimes |\gamma_i\rangle_C)(\langle\chi_i|_{AB} \otimes \langle\gamma_i|_C) \\
&= \sum_i \lambda_i (|\chi_i\rangle_{AB} \langle\chi_i|_{AB}) \otimes (|\gamma_i\rangle_C \langle\gamma_i|_C) \\
&= \sum_i \lambda_i \rho_{AB}^{(i)} \otimes \rho_C^{(i)}. \qquad (4.2.4)
\end{aligned}
$$

One can easily work out what it means to be A:BC or B:AC separable. The part that is different for tripartite mixed states is that ρ_{ABC} is also said to be bi-separable (without considering partitions) as long as there exists a decomposition where ρ_{ABC} is made up of bi-product states.

To illustrate, let us denote $\rho_{A:BC}$, $\rho_{B:AC}$ and $\rho_{AB:C}$ where the subscripts indicate the partitions they are separable in. Then ρ_{ABC} is bi-separable (without considering partitions) if it is made of a linear combination of these bi-product states,

$$\rho_{ABC} = \lambda_1 \rho_{A:BC} + \lambda_2 \rho_{B:AC} + \lambda_3 \rho_{AB:C}, \qquad (4.2.5)$$

and λ_1, λ_2 and λ_3 are probabilities.

4.3 Relative entropy of entanglement

The relative entropy of entanglement (REE) is a measure that quantifies entanglement for general states [13] . It is defined as

$$E(\rho) = \min_{\sigma' \in \mathcal{D}} S(\rho||\sigma') = -\mathrm{tr}(\rho \log(\sigma)) + \mathrm{tr}(\rho \log(\rho)), \qquad (4.3.1)$$

where σ is the closest separable state to ρ. It is important to specify the partition for $\mathcal{D}$. For instance, if we talk about the partition $AB : C$, then $\mathcal{D}$ refers to the set of $AB : C$ separable states and we perform the minimization over all such $AB : C$ separable states, σ'. We denote σ to be the state satisfying the minimum, hence it is the closest $AB : C$ separable state to ρ.

A visual representation is given in Figure 4.1.

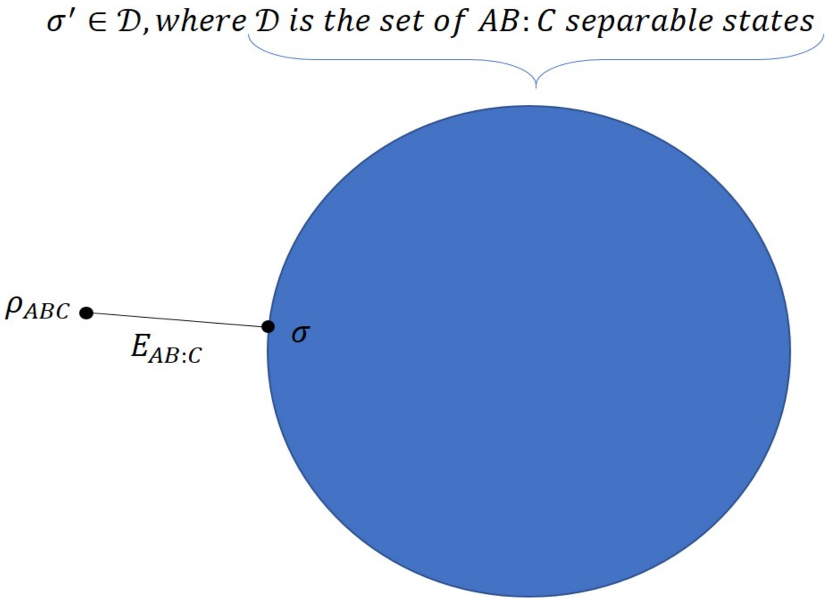

Figure 4.1: Visualization of the relative entropy of entanglement for the $AB : C$ partition.

For pure states, REE was found to be equivalent to the von Neumann entropy of subsystems in the relevant partitions [14]. In other words we can write,

$$\begin{aligned} E_{AB:C} &= S_{AB} = S_C, \\ E_{A:BC} &= S_{BC} = S_A, \\ E_{AC:B} &= S_{AC} = S_B. \end{aligned} \tag{4.3.2}$$

Minimization is not a computationally easy task. In this thesis, we will be dealing with pure states so we will use the latter instead.

Chapter 5

Our findings

Entanglement between A and B can be created via various methods, one of which is allowing A and B to continuously interact with an ancilla, C. In this thesis, we focus on qubits, i.e. the dimension of A, B, and C are 2 each.

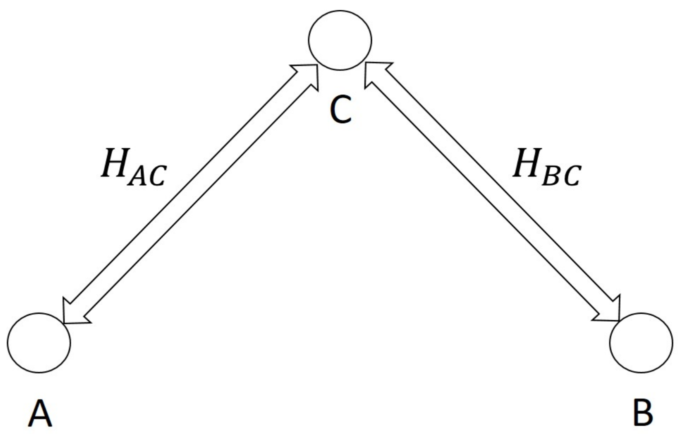

Figure 5.1: Setup for distributing entanglement between A and B via continuous interaction with C.

In Section 5.1, we look at H_{AC} and H_{BC} that take the form of general Hamiltonians. We ask the question, what does it mean for H_{AC} and H_{BC} to commute? We investigate this in Theorem 2 and we show two things. One, that there is a common eigenbasis on C, and two, we are able to rewrite $H_{AC} = H_A + \tilde{H}_A \otimes \tilde{H}_C$ and $H_{BC} = H_B + \tilde{H}_B \otimes \tilde{H}_C$ (for simplicity, we shall call this subclass of Hamiltonians $\mathcal{H}_1$).

In Section 5.2, we introduce the case where S_{A-C} is the unitary that swaps the states of A with C. We construct the Hamiltonian H_{AC} that generates S_{A-C} at a particular time, and H_{BC} is the identity operator (for simplicity, we shall call this particular pair of Hamiltonians $\mathcal{H}_2$).

In Section 5.2.1, we introduce the idea of using the Schmidt decomposition on

$$\tilde{H}_{AC} = \frac{\pi}{4}\sigma_1 \otimes \mathbb{1} \otimes \sigma_1 + \frac{\pi}{4}\sigma_2 \otimes \mathbb{1} \otimes \sigma_2 + \frac{\pi}{4}\sigma_3 \otimes \mathbb{1} \otimes \sigma_3, \tag{5.0.1}$$

and we show that there is no way to write it as a single product ($\tilde{H}_{AC} \neq \tilde{H}_A \otimes \tilde{H}_C$).

In Section 5.3, we investigate the time evolution of initial states $|\alpha\rangle_A |\beta\rangle_B |\gamma\rangle_C$ via interaction Hamiltonians $\mathcal{H}_1$. We find at any time, $S_A(t) \leq S_C(t)$ and $S_B(t) \leq S_C(t)$, and the analytical proof of this is shown in Theorem 4.

In Section 5.4, we restrict ourselves to pure bi-product states of the form $|\psi_0\rangle = |\chi\rangle_{AB} |\gamma\rangle_C$. First, we consider interaction Hamiltonians $\mathcal{H}_1$. In MATLAB, we generated 100 random pairs of such Hamiltonians and checked each one for 10,000 random bi-product states $|\chi\rangle_{AB} |\gamma\rangle_C$. Our numerical results suggests the following bound:

$$\begin{aligned} S_A(t) - S_A(0) &\leq S_C(t), \\ S_B(t) - S_B(0) &\leq S_C(t). \end{aligned} \tag{5.0.2}$$

We find this bound intuitive; since we start with entanglement between A and B, it is not surprising that we obtain a modification of Theorem 4 that takes care of the initial entanglement between A and B. However we have not been able to prove this bound analytically. We present a plot that shows the tightness of this bound for some random $|\chi\rangle_{AB} |\gamma\rangle_C$. Later in Section 5.4, we consider the time evolution given by interaction Hamiltonians $\mathcal{H}_2$. We find

$$|S_A(t) - S_A(0)| \leq S_C(t), \tag{5.0.3}$$

and we show the analytical proof in Theorem 5. Here, we also show that if A and B were initially product states, $S_A(t)$ and $S_C(t)$ would be equal.

In the last part of Section 5.4, we return to interaction Hamiltonians $\mathcal{H}_1$. We prove in Theorem 6 that if A and B are initially entangled ($|\chi\rangle_{AB} |\gamma\rangle_C$), local unitary dynamics kicks in and we find $S_A(t) \geq S_A(0)$ and $S_B(t) \geq S_B(0)$. We use Theorem 6 to show that $S_A(0) - S_A(t) \leq S_C(t)$ and $S_B(0) - S_B(t) \leq S_C(t)$.

5.1 Commuting Hamiltonians

When we refer to the general Hamiltonian H_{AC}, implicitly we mean that all tensor products with system B involve the identity operator on B:

$$H_{AC} = \sum_{a=0}^{3} \sum_{c=0}^{3} h_{ac} \sigma_a \otimes \mathbb{1} \otimes \sigma_c$$

Also, we want A and C to interact, so we do not want H_{AC} to be just H_A (free Hamiltonians on A) or H_C (free Hamiltonians on C). Similarly, for general Hamiltonians H_{BC}, implicitly we mean that all tensor products with A involve the identity operator on A.

$$H_{BC} = \sum_{b=0}^{3} \sum_{c'=0}^{3} f_{bc'} \mathbb{1} \otimes \sigma_b \otimes \sigma_{c'}$$

Since we want B and C to interact as well, we do not want H_{BC} to be just H_B (free Hamiltonians on B) or H_C (free Hamiltonians on C).

In this Section, we present Theorem 2 that allows us to simplify H_{AC} and H_{BC} if they are commuting. Lemma 1 will be useful in proving Theorem 2.

Lemma 1. *For any diagonalizable square matrix M, if M' is proportional to M and different proportions of identity are added to both M and M', the resulting matrices still share a set of common eigenbasis.*

Proof. Suppose M is some $m \times m$ square matrix of dimensionality m. Since M is diagonalizable, there exists an eigendecomposition of M which reads,

$$M = \sum_{n=1}^{m} \lambda_n \left|\lambda_n\right\rangle \left\langle\lambda_n\right|, \tag{5.1.1}$$

where $\{\lambda_1, \lambda_2, ..., \lambda_m\}$ are the eigenvalues and $\{\left|\lambda_1\right\rangle, \left|\lambda_2\right\rangle, ..., \left|\lambda_m\right\rangle\}$ are the eigenvectors of M. Since M' is proportional to M, naturally they share the same eigenvectors, just that the eigenvalues of M' will be scaled by some proportionality factor.

$$M' = \sum_{n=1}^{m} \lambda'_n \left|\lambda_n\right\rangle \left\langle\lambda_n\right|, \tag{5.1.2}$$

where $\{\lambda'_1, \lambda'_2, ..., \lambda'_m\}$ are the scaled eigenvalues. Since $\{\left|\lambda_1\right\rangle, \left|\lambda_2\right\rangle, ..., \left|\lambda_m\right\rangle\}$ arises from the eigendecomposition, these eigenvectors form a complete orthonormal basis. The identity matrix can be expressed using that set of orthonormal vectors,

$$\mathbb{1}_m = \sum_{n=1}^{m} \left|\lambda_n\right\rangle \left\langle\lambda_n\right|. \tag{5.1.3}$$

Adding different proportions of identity to M and M' gives us

$$\begin{aligned} M + \alpha\mathbb{1}_m &= \sum_{n=1}^{m} (\alpha + \lambda_n) \left|\lambda_n\right\rangle \left\langle\lambda_n\right|, \\ M' + \beta\mathbb{1}_m &= \sum_{n=1}^{m} (\beta + \lambda'_n) \left|\lambda_n\right\rangle \left\langle\lambda_n\right|, \end{aligned} \tag{5.1.4}$$

where α and β are different constants. One can see that the resultant matrices share a set of common eigenbasis $\{\left|\lambda_1\right\rangle, \left|\lambda_2\right\rangle, ..., \left|\lambda_m\right\rangle\}$. □

When two operators commute, it is known that they share a common eigenbasis. However in general, this common eigenbasis can be entangled. In Theorem 2, we show that if $[H_{AC}, H_{BC}] = 0$, then the common eigenbasis is *product* on C. This simplification will be applied to other Theorems later on.

Theorem 2. *Consider general Hamiltonians*

$$H_{AC} = \sum_{a=0}^{3}\sum_{c=0}^{3} h_{ac}\sigma_a \otimes \mathbb{1} \otimes \sigma_c, \tag{5.1.5}$$

$$H_{BC} = \sum_{b=0}^{3}\sum_{c'=0}^{3} f_{bc'}\mathbb{1} \otimes \sigma_b \otimes \sigma_{c'}, \tag{5.1.6}$$

where $\sigma_0 = \mathbb{1}$ and $\sigma_{i=1,2,3}$ are the 2×2 Pauli matrices. The total Hamiltonian is given by $H = H_{AC} + H_{BC}$. If $[H_{AC}, H_{BC}] = 0$, and all the coefficients h_{ac} and $f_{bc'}$ are non-vanishing, then H_{AC} and H_{BC} can always be rewritten as

$$H_{AC} = H_A + \tilde{H}_A \otimes \tilde{H}_C, \tag{5.1.7}$$

$$H_{BC} = H_B + \tilde{H}_B \otimes \tilde{H}_C, \tag{5.1.8}$$

and there exists a set of common eigenbasis on C.

Proof. First let us consider a simpler case where the sum over a only has two terms, a and a', and the sum over b also has two terms, b and b'. The Hamiltonians H_{AC} and H_{BC} read:

$$H_{AC} = \sigma_a \otimes \mathbb{1} \otimes \sum_{c=0}^{3} h_{ac}\sigma_c + \sigma_{a'} \otimes \mathbb{1} \otimes \sum_{c=0}^{3} h_{a'c}\sigma_c, \tag{5.1.9}$$

$$H_{BC} = \mathbb{1} \otimes \sigma_b \otimes \sum_{c'=0}^{3} f_{bc'}\sigma_{c'} + \mathbb{1} \otimes \sigma_{b'} \otimes \sum_{c'=0}^{3} f_{b'c'}\sigma_{c'}. \tag{5.1.10}$$

We can take out σ_0 and the coefficient tagged to it. Rewriting Eq.(5.1.9),

$$\begin{aligned} H_{AC} &= \sigma_a \otimes \mathbb{1} \otimes \left(h_{a0}\mathbb{1} + \sum_{c=1}^{3} h_{ac}\sigma_c \right) + \sigma_{a'} \otimes \mathbb{1} \otimes \left(h_{a'0}\mathbb{1} + \sum_{c=1}^{3} h_{a'c}\sigma_c \right) \\ &= (h_{a0}\sigma_a + h_{a'0}\sigma_{a'}) \otimes \mathbb{1} \otimes \mathbb{1} + \sigma_a \otimes \mathbb{1} \otimes \sum_{c=1}^{3} h_{ac}\sigma_c + \sigma_{a'} \otimes \mathbb{1} \otimes \sum_{c=1}^{3} h_{a'c}\sigma_c \qquad (5.1.11) \\ &= H_A + \tilde{H}_{AC}, \qquad (5.1.12) \end{aligned}$$

where $H_A = (h_{a0}\sigma_a + h_{a'0}\sigma_{a'}) \otimes \mathbb{1} \otimes \mathbb{1}$ and $\tilde{H}_{AC}$ comprises of the remaining two terms.

Rewriting Eq.(5.1.10),

$$
\begin{aligned}
H_{BC} &= \mathbb{1} \otimes \sigma_b \otimes \left(f_{b0}\mathbb{1} + \sum_{c'=1}^{3} f_{bc'}\sigma_{c'} \right) + \mathbb{1} \otimes \sigma_{b'} \otimes \left(f_{b'0}\mathbb{1} + \sum_{c'=1}^{3} f_{b'c'}\sigma_{c'} \right) \\
&= \mathbb{1} \otimes (f_{b0}\sigma_b + f_{b'0}\sigma_{b'}) \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_b \otimes \sum_{c'=1}^{3} f_{bc'}\sigma_{c'} + \mathbb{1} \otimes \sigma_{b'} \otimes \sum_{c'=1}^{3} f_{b'c'}\sigma_{c'} \qquad (5.1.13) \\
&= H_B + \tilde{H}_{BC}, \qquad (5.1.14)
\end{aligned}
$$

where $H_B = \mathbb{1} \otimes (f_{b0}\sigma_b + f_{b'0}\sigma_{b'}) \otimes \mathbb{1}$ and $\tilde{H}_{BC}$ comprises of the remaining two terms.

By doing so, we have split H_{AC} into a part that only acts on A, and a part that provides continuous interaction between A and C. Likewise for H_{BC}. Then the condition, $[H_{AC}, H_{BC}] = 0$ becomes

$$
\begin{aligned}
0 &= [H_{AC}, H_{BC}] \\
&= \left[H_A + \tilde{H}_{AC}, H_B + \tilde{H}_{BC} \right] \\
&= [H_A, H_B] + \left[H_A, \tilde{H}_{BC} \right] + \left[\tilde{H}_{AC}, H_B \right] + \left[\tilde{H}_{AC}, \tilde{H}_{BC} \right] \\
&\mid \text{ It turns out that } [H_A, H_B] = \left[H_A, \tilde{H}_{BC} \right] = \left[\tilde{H}_{AC}, H_B \right] = 0. \\
&= \left[\tilde{H}_{AC}, \tilde{H}_{BC} \right]. \qquad (5.1.15)
\end{aligned}
$$

H_A and H_B are the free Hamiltonians that give A and B their free energies. This is completely independent of the interaction Hamiltonians $\tilde{H}_{AC}$ and $\tilde{H}_{BC}$. Therefore wanting H_{AC} and H_{BC} to commute directly translates into the commutation between these “deeper” interaction Hamiltonians.

It is obvious from Eq. (5.1.11) and Eq. (5.1.13) that if we instead had the sum $a \in 0, 1, 2, 3$ and $b \in 0, 1, 2, 3$, our free Hamiltonians and interaction Hamiltonians would read:

$$
H_A = \sum_{a=0}^{3} h_{a0}\sigma_a \otimes \mathbb{1} \otimes \mathbb{1}, \; \tilde{H}_{AC} = \sum_{a=0}^{3} \sigma_a \otimes \mathbb{1} \otimes \sum_{c=1}^{3} h_{ac}\sigma_c, \qquad (5.1.16)
$$

$$
H_B = \mathbb{1} \otimes \sum_{b=0}^{3} f_{b0}\sigma_b \otimes \mathbb{1}, \; \tilde{H}_{BC} = \mathbb{1} \otimes \sum_{b=0}^{3} \sigma_b \otimes \sum_{c'=1}^{3} f_{bc'}\sigma_{c'}. \qquad (5.1.17)
$$

Expanding $\left[\tilde{H}_{AC}, \tilde{H}_{BC}\right] = 0$,

$$\begin{aligned}
0 &= \sum_{a=0}^{3} \sigma_a \otimes \sum_{b=0}^{3} \sigma_b \otimes \sum_{c=1}^{3}\sum_{c'=1}^{3} h_{ac} f_{bc'} \sigma_c \sigma_{c'} - \sum_{a=0}^{3} \sigma_a \otimes \sum_{b=0}^{3} \sigma_b \otimes \underbrace{\sum_{c'=1}^{3}\sum_{c=1}^{3} f_{bc'} h_{ac} \sigma_{c'} \sigma_c}_{(*)} \\
&\quad | \quad \text{Since } c \text{ and } c' \text{ both run from 1 to 3, we can interchange } c \text{ and } c' \text{ for (*).} \\
&= \sum_{a=0}^{3} \sigma_a \otimes \sum_{b=0}^{3} \sigma_b \otimes \sum_{c=1}^{3}\sum_{c'=1}^{3} h_{ac} f_{bc'} \sigma_c \sigma_{c'} - \sum_{a=0}^{3} \sigma_a \otimes \sum_{b=0}^{3} \sigma_b \otimes \sum_{c=1}^{3}\sum_{c'=1}^{3} f_{bc} h_{ac'} \sigma_c \sigma_{c'} \\
&= \sum_{a=0}^{3} \sigma_a \otimes \sum_{b=0}^{3} \sigma_b \otimes \sum_{c=1}^{3}\sum_{c'=1}^{3} (h_{ac} f_{bc'} - f_{bc} h_{ac'}) \sigma_c \sigma_{c'}. \qquad (5.1.18)
\end{aligned}$$

The Pauli matrices and identity make up a linearly independent set, so the only way for Eq.(5.1.18) to vanish is for every coefficient in front of distinct tensor products of Pauli matrices to vanish. Rewriting Eq.(5.1.18),

$$\sum_{a=0}^{3}\sum_{b=0}^{3}\sum_{c=1}^{3}\sum_{c'=1}^{3} (h_{ac} f_{bc'} - f_{bc} h_{ac'}) \sigma_a \otimes \sigma_b \otimes \sigma_c \sigma_{c'} = 0 \qquad (5.1.19)$$

$$\sum_{a=0}^{3}\sum_{b=0}^{3}\sum_{c=1}^{3}\sum_{c'=1}^{3} (h_{ac} f_{bc'} - f_{bc} h_{ac'}) \sigma_a \otimes \sigma_b \otimes (\delta_{cc'} \mathbb{1} + i\epsilon_{cc'k} \sigma_k) = 0, \qquad (5.1.20)$$

where $\epsilon_{cc'k}$ is the Levi-Civita symbol.

Pauli matrices are unitary and Hermitian. Acting two of the same Pauli matrices gives us identity, therefore terms proportional to identity corresponds to cases where $c = c'$. However when $c = c'$, the coefficients cancel each other so there are no terms proportional to identity.

For $c \neq c'$, we know that the product of two Pauli matrices yields in another Pauli matrix in a cyclic order. First, let us look at terms which contribute to $\sigma_c \sigma_c'$ being proportional to σ_1. These correspond to

$$\begin{aligned}
\sigma_2 \sigma_3 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\sigma_3 \sigma_2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \qquad (5.1.21)
\end{aligned}$$

For the coefficient in front of σ_1 to vanish, we need the coefficients of these two contributions to sum up to zero:

$$i(h_{a2}f_{b3} - f_{b2}h_{a3}) - i(h_{a3}f_{b2} - f_{b3}h_{a2}) = 0, \tag{5.1.22}$$

$$2i(h_{a2}f_{b3} - f_{b2}h_{a3}) = 0 \Rightarrow \frac{h_{a2}}{f_{b2}} = \frac{h_{a3}}{f_{b3}}. \tag{5.1.23}$$

Repeating the same steps for terms proportional to σ_2 and σ_3, one will find respectively,

$$\begin{aligned} 2i(h_{a3}f_{b1} - f_{b3}h_{a1}) = 0 \Rightarrow \frac{h_{a1}}{f_{b1}} = \frac{h_{a3}}{f_{b3}}, \\ 2i(h_{a1}f_{b2} - f_{b1}h_{a2}) = 0 \Rightarrow \frac{h_{a1}}{f_{b1}} = \frac{h_{a2}}{f_{b2}}. \end{aligned} \tag{5.1.24}$$

Combining these results, we obtain the set of relations that set $\left[\tilde{H}_{AC}, \tilde{H}_{BC}\right]$ to zero:

$$\frac{h_{a1}}{f_{b1}} = \frac{h_{a2}}{f_{b2}} = \frac{h_{a3}}{f_{b3}}. \tag{5.1.25}$$

Since Pauli matrices are linearly independent and they form a basis, one can treat these relations as a form of matrix parallelism. Suppose we consider two vectors in 2D space, where x and y form a basis. The two vectors are said to be parallel if $x_1/x_2 = y_1/y_2$. This is analogous to our relation in Eq.(5.1.25).

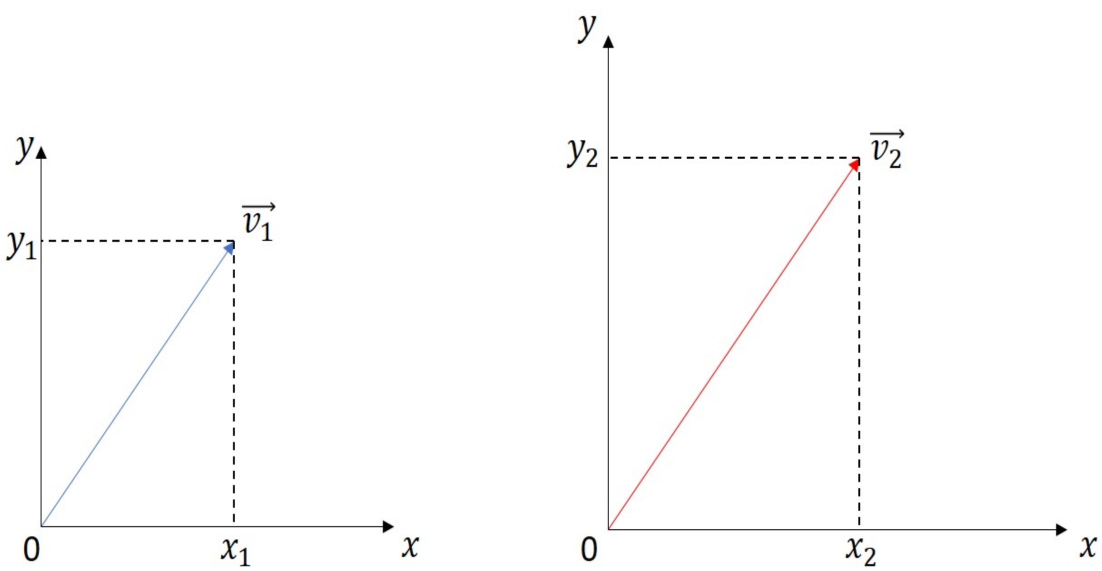

Figure 5.2: The common ratio of coefficients is analogous to that of parallel vectors.

Since $a, b \in 0, 1, 2, 3$, there will be 16 sets of these relations pertaining to each a and b:

$$\begin{aligned}
&\left.\frac{h_{01}}{f_{01}} = \frac{h_{02}}{f_{02}} = \frac{h_{03}}{f_{03}}\right\} a = 0,\ b = 0 \qquad &&\left.\frac{h_{01}}{f_{11}} = \frac{h_{02}}{f_{12}} = \frac{h_{03}}{f_{13}}\right\} a = 0,\ b = 1 \\
&\left.\frac{h_{11}}{f_{01}} = \frac{h_{12}}{f_{02}} = \frac{h_{13}}{f_{03}}\right\} a = 1,\ b = 0 \qquad &&\left.\frac{h_{11}}{f_{11}} = \frac{h_{12}}{f_{12}} = \frac{h_{13}}{f_{13}}\right\} a = 1,\ b = 1 \\
&\left.\frac{h_{21}}{f_{01}} = \frac{h_{22}}{f_{02}} = \frac{h_{23}}{f_{03}}\right\} a = 2,\ b = 0 \qquad &&\left.\frac{h_{21}}{f_{11}} = \frac{h_{22}}{f_{12}} = \frac{h_{23}}{f_{13}}\right\} a = 2,\ b = 1 \\
&\left.\frac{h_{31}}{f_{01}} = \frac{h_{32}}{f_{02}} = \frac{h_{33}}{f_{03}}\right\} a = 3,\ b = 0 \qquad &&\left.\frac{h_{31}}{f_{11}} = \frac{h_{32}}{f_{12}} = \frac{h_{33}}{f_{13}}\right\} a = 3,\ b = 1 \\
&\left.\frac{h_{01}}{f_{21}} = \frac{h_{02}}{f_{22}} = \frac{h_{03}}{f_{23}}\right\} a = 0,\ b = 2 \qquad &&\left.\frac{h_{01}}{f_{31}} = \frac{h_{02}}{f_{32}} = \frac{h_{03}}{f_{33}}\right\} a = 0,\ b = 3 \\
&\left.\frac{h_{11}}{f_{21}} = \frac{h_{12}}{f_{22}} = \frac{h_{13}}{f_{23}}\right\} a = 1,\ b = 2 \qquad &&\left.\frac{h_{11}}{f_{31}} = \frac{h_{12}}{f_{32}} = \frac{h_{13}}{f_{33}}\right\} a = 1,\ b = 3 \\
&\left.\frac{h_{21}}{f_{21}} = \frac{h_{22}}{f_{22}} = \frac{h_{23}}{f_{23}}\right\} a = 2,\ b = 2 \qquad &&\left.\frac{h_{21}}{f_{31}} = \frac{h_{22}}{f_{32}} = \frac{h_{23}}{f_{33}}\right\} a = 2,\ b = 3 \\
&\left.\frac{h_{31}}{f_{21}} = \frac{h_{32}}{f_{22}} = \frac{h_{33}}{f_{23}}\right\} a = 3,\ b = 2 \qquad &&\left.\frac{h_{31}}{f_{31}} = \frac{h_{32}}{f_{32}} = \frac{h_{33}}{f_{33}}\right\} a = 3,\ b = 3
\end{aligned} \tag{5.1.26}$$

Since we are working under the condition that these coefficients are non-vanishing, Eq.(5.1.26) really brings out the idea of matrix parallelism. Let us look at the terms pertaining to $\{a = 0, b = 0\}$. Our original Hamiltonians read,

$$H_{AC}^{(a=0)} = \sigma_0 \otimes \mathbb{1} \otimes (h_{00}\mathbb{1} + h_{01}\sigma_1 + h_{02}\sigma_2 + h_{03}\sigma_3), \tag{5.1.27}$$

$$H_{BC}^{(b=0)} = \mathbb{1} \otimes \sigma_0 \otimes (f_{00}\mathbb{1} + f_{01}\sigma_1 + f_{02}\sigma_2 + f_{03}\sigma_3). \tag{5.1.28}$$

The set of relations for $\{a = 0, b = 0\}$ in Eq. (5.1.26) implies proportional matrices:

$$\begin{aligned}
h_{01}\sigma_1 + h_{02}\sigma_2 + h_{03}\sigma_3 &= \begin{pmatrix} h_{03} & h_{01} - ih_{02} \\ h_{01} + ih_{02} & -h_{03} \end{pmatrix} \\
&= \frac{h_{01}}{f_{01}} \begin{pmatrix} f_{03} & f_{01} - if_{02} \\ f_{01} + if_{02} & -f_{03} \end{pmatrix} \\
&= \frac{h_{01}}{f_{01}} (f_{01}\sigma_1 + f_{02}\sigma_2 + f_{03}\sigma_3).
\end{aligned} \tag{5.1.29}$$

We can introduce a new matrix, σ_s, which reads

$$\sigma_s = \frac{1}{f_{01}} (f_{01}\sigma_1 + f_{02}\sigma_2 + f_{03}\sigma_3), \tag{5.1.30}$$

then Eq.(5.1.27) and Eq.(5.1.28) can be rewritten as

$$H_{AC}^{(a=0)} = \sigma_0 \otimes \mathbb{1} \otimes (h_{00}\mathbb{1} + h_{01}\sigma_s), \tag{5.1.31}$$

$$H_{BC}^{(b=0)} = \mathbb{1} \otimes \sigma_0 \otimes (f_{00}\mathbb{1} + f_{01}\sigma_s). \tag{5.1.32}$$

Comparing Eq.(5.1.31) and Eq.(5.1.32), we see that we can apply Lemma 1 to the terms in the brackets. There exists a set of common eigenbasis on C; the eigenbasis of σ_s. Repeating this for all $a, b \in 0, 1, 2, 3$, we find

$$\begin{aligned}
H_{AC}^{(a=0)} &= \sigma_0 \otimes \mathbb{1} \otimes (h_{00}\mathbb{1} + h_{01}\sigma_s) \qquad & H_{BC}^{(b=0)} &= \mathbb{1} \otimes \sigma_0 \otimes (f_{00}\mathbb{1} + f_{01}\sigma_s)\\
H_{AC}^{(a=1)} &= \sigma_1 \otimes \mathbb{1} \otimes (h_{10}\mathbb{1} + h_{11}\sigma_s) \qquad & H_{BC}^{(b=1)} &= \mathbb{1} \otimes \sigma_1 \otimes (f_{10}\mathbb{1} + f_{11}\sigma_s)\\
H_{AC}^{(a=2)} &= \sigma_2 \otimes \mathbb{1} \otimes (h_{20}\mathbb{1} + h_{21}\sigma_s) \qquad & H_{BC}^{(b=2)} &= \mathbb{1} \otimes \sigma_2 \otimes (f_{20}\mathbb{1} + f_{21}\sigma_s)\\
H_{AC}^{(a=3)} &= \sigma_3 \otimes \mathbb{1} \otimes (h_{30}\mathbb{1} + h_{31}\sigma_s) \qquad & H_{BC}^{(b=3)} &= \mathbb{1} \otimes \sigma_3 \otimes (f_{30}\mathbb{1} + f_{31}\sigma_s).
\end{aligned} \tag{5.1.33}$$

It is now obvious that there is indeed a common eigenbasis on C. Rewriting Eq.(5.1.33), we find that

$$\begin{aligned}
\tilde{H}_{AC} &= (h_{01}\sigma_0 + h_{11}\sigma_1 + h_{21}\sigma_2 + h_{31}\sigma_3) \otimes \mathbb{1} \otimes \sigma_s\\
&= \sum_{a=0}^{3} h_{a1}\sigma_a \otimes \mathbb{1} \otimes \sigma_s\\
&= \tilde{H}_A \otimes \tilde{H}_C, \qquad (5.1.34)\\
\tilde{H}_{BC} &= \mathbb{1} \otimes (f_{01}\sigma_0 + f_{11}\sigma_1 + f_{21}\sigma_2 + f_{31}\sigma_3) \otimes \sigma_s\\
&= \mathbb{1} \otimes \sum_{b=0}^{3} h_{b1}\sigma_b \otimes \sigma_s\\
&= \tilde{H}_B \otimes \tilde{H}_C, \qquad (5.1.35)
\end{aligned}$$

where $\tilde{H}_A = \sum_{a=0}^{3} h_{a1}\sigma_a$, $\tilde{H}_B = \sum_{b=0}^{3} f_{b1}\sigma_b$ and $\tilde{H}_C = \sigma_s$.

Therefore the H_{AC} and H_{BC} can be rewritten as

$$\begin{aligned}
H_{AC} &= H_A + \tilde{H}_A \otimes \tilde{H}_C\\
H_{BC} &= H_B + \tilde{H}_B \otimes \tilde{H}_C.
\end{aligned} \tag{5.1.36}$$

This completes the proof. □

Theorem 2 tells us that if we want to create random commuting general Hamiltonians, instead of meticulously calibrating all 16 sets of relations in Eq.(5.1.26), we can just define σ_s and set it up. If all the coefficients are non-vanishing, σ_s is just given by Eq.(5.1.30) and H_{AC} and H_{BC} can be constructed as follows:

$$\begin{aligned} H_{AC} &= (h_{00}\sigma_0 + h_{10}\sigma_1 + h_{20}\sigma_2 + h_{30}\sigma_3) \otimes \mathbb{1} \otimes \mathbb{1} + (h_{01}\sigma_0 + h_{11}\sigma_1 + h_{21}\sigma_2 + h_{31}\sigma_3) \otimes \mathbb{1} \otimes \sigma_s, \\ H_{BC} &= \mathbb{1} \otimes (f_{00}\sigma_0 + f_{10}\sigma_1 + f_{20}\sigma_2 + f_{30}\sigma_3) \otimes \mathbb{1} + \mathbb{1} \otimes (f_{01}\sigma_0 + f_{11}\sigma_1 + f_{21}\sigma_2 + f_{31}\sigma_3) \otimes \sigma_s. \end{aligned} \tag{5.1.37}$$

All these coefficients and σ_s can be randomed (as there is no relation in Eq.(5.1.26) relating these coefficients together). This is how we construct random general Hamiltonians that commute (in MATLAB) to test our theorems.

For the case where some coefficients h_{ac} and $f_{bc'}$ are vanishing, to show that Theorem 2 holds becomes hairy problem that is not easy to deal with. There are too many permutations and possible cases to consider and we will not delve into the details in this thesis.

5.2 H_{AC}: Swap operator, H_{BC}: Identity operator

In this Section, we introduce the Hamiltonian that generates the swap operator which state of A with C. Later on, we will look at the evolution of tripartite pure states using such Hamiltonians. The unitary that performs the swap, S_{A-C}, can be constructed as follows:

$$
\begin{aligned}
|000\rangle \rightarrow |000\rangle \Big\} & (|000\rangle \langle 000|) |000\rangle = |000\rangle \\
|001\rangle \rightarrow |100\rangle \Big\} & (|100\rangle \langle 001|) |001\rangle = |100\rangle \\
|010\rangle \rightarrow |010\rangle \Big\} & (|010\rangle \langle 010|) |010\rangle = |010\rangle \\
|011\rangle \rightarrow |110\rangle \Big\} & (|110\rangle \langle 011|) |011\rangle = |110\rangle \\
|100\rangle \rightarrow |001\rangle \Big\} & (|001\rangle \langle 100|) |100\rangle = |001\rangle \\
|101\rangle \rightarrow |101\rangle \Big\} & (|101\rangle \langle 101|) |101\rangle = |101\rangle \\
|110\rangle \rightarrow |011\rangle \Big\} & (|011\rangle \langle 110|) |110\rangle = |011\rangle \\
|111\rangle \rightarrow |111\rangle \Big\} & (|111\rangle \langle 111|) |111\rangle = |111\rangle
\end{aligned}
$$

The corresponding matrix representation for unitary that performs the swap is:

$$
S_{A-C} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}, \tag{5.2.1}
$$

with ones in the relevant places. It turns out that an equivalent representation for S_{A-C} is

$$
S_{A-C} = \frac{1}{2}\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} + \frac{1}{2}\sigma_1 \otimes \mathbb{1} \otimes \sigma_1 + \frac{1}{2}\sigma_2 \otimes \mathbb{1} \otimes \sigma_2 + \frac{1}{2}\sigma_3 \otimes \mathbb{1} \otimes \sigma_3. \tag{5.2.2}
$$

Note that the the unitary that swaps the states of B and C is written as

$$
S_{B-C} = \frac{1}{2}\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} + \frac{1}{2}\mathbb{1} \otimes \sigma_1 \otimes \sigma_1 + \frac{1}{2}\mathbb{1} \otimes \sigma_2 \otimes \sigma_2 + \frac{1}{2}\mathbb{1} \otimes \sigma_3 \otimes \sigma_3, \tag{5.2.3}
$$

and $[S_{A-C}, S_{B-C}] \neq 0$. This makes sense. If we start with $|\alpha\rangle_A |\beta\rangle_B |\gamma\rangle_C$ and swap B with C first, we have $|\alpha\rangle_A |\gamma\rangle_B |\beta\rangle_C$. Then we swap A with C and end up with $|\beta\rangle_A |\gamma\rangle_B |\alpha\rangle_C$. If we performed the S_{A-C} first followed by S_{B-C}, we would end up with $|\gamma\rangle_A |\alpha\rangle_B |\beta\rangle_C$.

We can find the eigenvalues and eigenvectors of S_{A-C} via MATLAB. It turns out that

$$S_{A-C} = (+1) \sum_{\mu=1}^{6} |\mu_+\rangle \langle\mu_+| + (-1) \sum_{\nu=1}^{2} |\nu_-\rangle \langle\nu_-| , \tag{5.2.4}$$

where $|\mu_+\rangle$ corresponds to the eigenvectors giving an eigenvalue of +1 and $|\nu_-\rangle$ corresponds to the eigenvectors giving an eigenvalue of -1. The relation between S_{A-C} and the Hamiltonian that generates it is $S_{A-C} = e^{-iH_{A-C}}$. Taking the natural logarithm on both sides, and using the fact that the logarithm of a matrix is just the logarithm of its eigenvalues in eigendecomposition,

$$\ln\left((+1) \sum_{\mu=1}^{6} |\mu_+\rangle \langle\mu_+| + (-1) \sum_{\nu=1}^{2} |\nu_-\rangle \langle\nu_-| \right) = -iH_{A-C} \tag{5.2.5}$$

$$\ln(+1) \sum_{\mu=1}^{6} |\mu_+\rangle \langle\mu_+| + \ln\left(e^{i\pi}\right) \sum_{\nu=1}^{2} |\nu_-\rangle \langle\nu_-| = -iH_{A-C} \tag{5.2.6}$$

$$i\pi \sum_{\nu=1}^{2} |\nu_-\rangle \langle\nu_-| = -iH_{A-C} \tag{5.2.7}$$

$$H_{A-C} = -\pi \sum_{\nu=1}^{2} |\nu_-\rangle \langle\nu_-| \tag{5.2.8}$$

It turns out that

$$\begin{aligned} |1_-\rangle &= \frac{1}{\sqrt{2}}(|0\rangle_A |0\rangle_B |1\rangle_C - |1\rangle_A |0\rangle_B |0\rangle_C) \\ &= \frac{1}{\sqrt{2}}(|0\rangle_A |1\rangle_C - |1\rangle_A |0\rangle_C) |0\rangle_B \\ &= \left|\Psi^-\right\rangle_{AC} |0\rangle_B , \end{aligned} \tag{5.2.9}$$

$$\begin{aligned} |2_-\rangle &= \frac{1}{\sqrt{2}}(|0\rangle_A |1\rangle_B |1\rangle_C - |1\rangle_A |1\rangle_B |0\rangle_C) \\ &= \frac{1}{\sqrt{2}}(|0\rangle_A |1\rangle_C - |1\rangle_A |0\rangle_C) |1\rangle_B \\ &= \left|\Psi^-\right\rangle_{AC} |1\rangle_B , \end{aligned} \tag{5.2.10}$$

where $|\Psi^-\rangle$ is a Bell state.

Rewriting Eq.(5.2.8),

$$\begin{aligned} H_{A-C} &= -\pi(|1_-\rangle\langle 1_-| + |2_-\rangle\langle 2_-|) \\ &= -\pi\big(|\Psi^-\rangle_{AC}\langle\Psi^-| \otimes |0\rangle_B\langle 0| + |\Psi^-\rangle_{AC}\langle\Psi^-| \otimes |1\rangle_B\langle 1|\big) \\ &= -\pi\big(|\Psi^-\rangle_{AC}\langle\Psi^-| \otimes \mathbb{1}_B\big). \end{aligned} \tag{5.2.11}$$

We find that the above is equivalent to

$$H_{A-C} = -\frac{\pi}{4}(\mathbb{1}^A \otimes \mathbb{1}^C - \sigma_1^A \otimes \sigma_1^C - \sigma_2^A \otimes \sigma_2^C - \sigma_3^A \otimes \sigma_3^C) \otimes \mathbb{1}^B, \tag{5.2.12}$$

where $\sigma_1, \sigma_2, \sigma_3$ are the x, y, z Pauli matrices. So how long does it take to swap? Let us put in units of energy into the Hamiltonian,

$$H_{AC} = \hbar\omega H_{A-C} = -\frac{\hbar\omega\pi}{4}(\mathbb{1}^A \otimes \mathbb{1}^C - \sigma_1^A \otimes \sigma_1^C - \sigma_2^A \otimes \sigma_2^C - \sigma_3^A \otimes \sigma_3^C) \otimes \mathbb{1}^B, \tag{5.2.13}$$

then using Eq.(5.2.13), we denote the unitary U_{AC} that generates the swap at a particular time,

$$U_{AC} = e^{-\frac{it}{\hbar}H_{AC}} = e^{-i\omega t H_{A-C}}. \tag{5.2.14}$$

When $t = 1/\omega$, U_{AC} becomes S_{A-C} and the swap is completed. If the frequency ω is high, it will take a shorter time to swap. In Theorem 5 we will see the time evolution of initial states $|\chi\rangle_{AB}|\gamma\rangle_C$ given by $H_{AC} = H_{A-C}$ (where we have set $\hbar = \omega = 1$) and $H_{BC} = \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}$.

Writing H_{A-C} in its diagonal representation,

$$H_{A-C} = \begin{bmatrix} -\pi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\pi & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.2.15}$$

we find

$$U_{AC} = e^{-i\omega t H_{A-C}} = \begin{bmatrix} e^{i\pi\omega t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{i\pi\omega t} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{5.2.16}$$

since the exponential of a diagonal matrix is just the exponential of the diagonal entries. Recalling that first two diagonal entries correspond to

$$e^{i\pi\omega t}(|1_-\rangle\langle 1_-| + |2_-\rangle\langle 2_-|) = e^{i\pi\omega t}\big(\left|\Psi^-\right\rangle_{AC}\left\langle\Psi^-\right| \otimes \mathbb{1}_B\big), \tag{5.2.17}$$

and the other entries correspond to $1(\mathbb{1}_A \otimes \mathbb{1}_B \otimes \mathbb{1}_C - |\Psi^-\rangle_{AC}\langle\Psi^-| \otimes \mathbb{1}_B)$, U_{AC} reads

$$\begin{aligned} U_{AC} &= 1(\mathbb{1}_A \otimes \mathbb{1}_B \otimes \mathbb{1}_C - \left|\Psi^-\right\rangle_{AC}\left\langle\Psi^-\right| \otimes \mathbb{1}_B) + e^{i\pi\omega t}(\left|\Psi^-\right\rangle_{AC}\left\langle\Psi^-\right| \otimes \mathbb{1}_B) \\ &= \mathbb{1}_A \otimes \mathbb{1}_B \otimes \mathbb{1}_C + (e^{i\pi\omega t} - 1)\left|\Psi^-\right\rangle_{AC}\left\langle\Psi^-\right| \otimes \mathbb{1}_B. \end{aligned} \tag{5.2.18}$$

As a quick check, we see that when $t = 0$, we are left with $U_{AC}(0) = \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}$. When $t = 1/\omega$,

$$\begin{aligned} U_{AC}(1/\omega) &= \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} - 2\big(\left|\Psi^-\right\rangle_{AC}\left\langle\Psi^-\right| \otimes \mathbb{1}_B\big) \\ &= \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} - 2\left[\frac{1}{4}(\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} - \sigma_1 \otimes \mathbb{1} \otimes \sigma_1 - \sigma_2 \otimes \mathbb{1} \otimes \sigma_2 - \sigma_3 \otimes \mathbb{1} \otimes \sigma_3)\right] \\ &= \frac{1}{2}\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} + \frac{1}{2}\sigma_1 \otimes \mathbb{1} \otimes \sigma_1 + \frac{1}{2}\sigma_2 \otimes \mathbb{1} \otimes \sigma_2 + \frac{1}{2}\sigma_3 \otimes \mathbb{1} \otimes \sigma_3, \end{aligned} \tag{5.2.19}$$

which is exactly Eq.(5.2.2).

5.2.1 Applying Schmidt Decomposition to Operators

From the Eq.(5.2.13), we had

$$H_{AC} = -\frac{\hbar\omega\pi}{4}\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} + \frac{\hbar\omega\pi}{4}\sigma_1 \otimes \mathbb{1} \otimes \sigma_1 + \frac{\hbar\omega\pi}{4}\sigma_2 \otimes \mathbb{1} \otimes \sigma_2 + \frac{\hbar\omega\pi}{4}\sigma_3 \otimes \mathbb{1} \otimes \sigma_3. \tag{5.2.20}$$

Let us denote

$$\tilde{H}_{AC} = \frac{\pi}{4}\sigma_1 \otimes \mathbb{1} \otimes \sigma_1 + \frac{\pi}{4}\sigma_2 \otimes \mathbb{1} \otimes \sigma_2 + \frac{\pi}{4}\sigma_3 \otimes \mathbb{1} \otimes \sigma_3, \tag{5.2.21}$$

where we have set $\hbar = \omega = 1$. We claim that $\tilde{H}_{AC}$ cannot be written as a product, $\tilde{H}_A \otimes \tilde{H}_C$. One can intuitively see that there is no way to decompose it into such a form, but we shall explicitly prove this and show that Eq.(5.2.21) is the Schmidt form.

Proof. Let us ignore the identity on B and write H_{AC} as

$$H_{AC} = \sum_{k=0}^{3}\sum_{l=0}^{3} h_{kl}\sigma_k \otimes \sigma_l. \tag{5.2.22}$$

Since Pauli matrices and identity form a basis, let us treat them as linearly independent vectors.

We define the following:

$$\sigma_0 = |0) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ \sigma_1 = |1) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ \sigma_2 = |2) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix},\ \sigma_3 = |3) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{5.2.23}$$

One should note that for $i \neq i'$,

$$\begin{aligned}(i|i') &= \mathrm{tr}(|i')(i|)\\ &= \mathrm{tr}(\sigma_{i'}\sigma_i)\\ &= \mathrm{tr}(i\epsilon_{i'ij}\sigma_j)\\ &= i\epsilon_{i'ij}\mathrm{tr}(\sigma_j)\\ &= 0,\end{aligned} \tag{5.2.24}$$

where $\epsilon_{i'ij}$ is the Levi-Civita symbol, and

$$\begin{aligned}(i|i) &= \mathrm{tr}(|i)(i|)\\ &= \mathrm{tr}(\sigma_i\sigma_i)\\ &= \mathrm{tr}(\sigma_0)\\ &= 2.\end{aligned} \tag{5.2.25}$$

We can rewrite Eq.(5.2.22) as

$$H_{AC} = \sum_{k=0}^{3} \sum_{l=0}^{3} h_{kl} |k)|l), \tag{5.2.26}$$

where the coefficients, h_{kl}, can be written into a column vector,

$$\mathcal{H}_{\text{vec}} = \begin{pmatrix} h_{00} \\ h_{01} \\ h_{02} \\ h_{03} \\ h_{10} \\ h_{11} \\ h_{12} \\ h_{13} \\ h_{20} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{30} \\ h_{31} \\ h_{32} \\ h_{33} \end{pmatrix}, \tag{5.2.27}$$

or a matrix,

$$\mathcal{H}_{\text{mat}} = \begin{pmatrix} h_{00} & h_{01} & h_{02} & h_{03} \\ h_{10} & h_{11} & h_{12} & h_{13} \\ h_{20} & h_{21} & h_{22} & h_{23} \\ h_{30} & h_{31} & h_{32} & h_{33} \end{pmatrix}. \tag{5.2.28}$$

By the Singular Value Decomposition, $\mathcal{H}_{\text{mat}} = UDV^\dagger$, and individual h_{kl}'s are given by $\sum_m U_{km} D_{mm} V^*_{lm}$. U and V are unitary square matrices and D is a diagonal matrix with real entries.

$\tilde{H}_{AC}$ then becomes

$$\tilde{H}_{AC} = \sum_{k,l,m} U_{km} D_{mm} V^*_{lm} |k)|l) = \sum_m D_{mm} \sum_k U_{km}|k) \sum_l V^*_{lm}|l) = \sum_m D_{mm}|\mu_m)|\nu_m), \quad (5.2.29)$$

where

$$|\mu_m) = \sum_{k=0}^{3} U_{km}|k) = U_{0m}|0) + U_{1m}|1) + U_{2m}|2) + U_{3m}|3), \quad (5.2.30)$$

$$|\nu_m) = \sum_{l=0}^{3} V^*_{lm}|l) = V^*_{0m}|0) + V^*_{1m}|1) + V^*_{2m}|2) + V^*_{3m}|3). \quad (5.2.31)$$

Like the usual Schmidt decomposition, $(\mu_m|\mu_{m'}) = (\nu_m|\nu_{m'})$ are zero. Since U is unitary, it is made up of orthogonal basis vectors,

$$|u_0\rangle = \begin{pmatrix} U_{00} \\ U_{10} \\ U_{20} \\ U_{30} \end{pmatrix}, \ |u_1\rangle = \begin{pmatrix} U_{01} \\ U_{11} \\ U_{21} \\ U_{31} \end{pmatrix}, \ |u_2\rangle = \begin{pmatrix} U_{02} \\ U_{12} \\ U_{22} \\ U_{32} \end{pmatrix}, \ |u_3\rangle = \begin{pmatrix} U_{03} \\ U_{13} \\ U_{23} \\ U_{33} \end{pmatrix}, \quad (5.2.32)$$

where $\langle u_i|u_j\rangle = 0$ for $i \neq j$. Since $\langle u_0|u_1\rangle = 0$, we have

$$U^*_{00}U_{01} + U^*_{10}U_{11} + U^*_{20}U_{21} + U^*_{30}U_{31} = 0. \quad (5.2.33)$$

We can calculate $(\mu_0|\mu_1)$ and find

$$\begin{aligned}
(\mu_0|\mu_1) &= \mathrm{tr}(|\mu_1)(\mu_0|) \\
&= \mathrm{tr}([U_{01}|0) + U_{11}|1) + U_{21}|2) + U_{31}|3)][U^*_{00}(0| + U^*_{10}(1| + U^*_{20}(2| + U^*_{30}(3|]) \\
&= [U_{01}U^*_{00}\mathrm{tr}(|0)(0|) + U_{01}U^*_{10}\mathrm{tr}(|0)(1|) + U_{01}U^*_{20}\mathrm{tr}(|0)(2|) + U_{01}U^*_{30}\mathrm{tr}(|0)(3|)] + \\
&\quad [U_{11}U^*_{00}\mathrm{tr}(|1)(0|) + U_{11}U^*_{10}\mathrm{tr}(|1)(1|) + U_{11}U^*_{20}\mathrm{tr}(|1)(2|) + U_{11}U^*_{30}\mathrm{tr}(|1)(3|)] + \\
&\quad [U_{21}U^*_{00}\mathrm{tr}(|2)(0|) + U_{21}U^*_{10}\mathrm{tr}(|2)(1|) + U_{21}U^*_{20}\mathrm{tr}(|2)(2|) + U_{21}U^*_{30}\mathrm{tr}(|2)(3|)] + \\
&\quad [U_{31}U^*_{00}\mathrm{tr}(|3)(0|) + U_{31}U^*_{10}\mathrm{tr}(|3)(1|) + U_{31}U^*_{20}\mathrm{tr}(|3)(2|) + U_{31}U^*_{30}\mathrm{tr}(|3)(3|)] \\
&= (U_{01}U^*_{00} + U_{11}U^*_{10} + U_{21}U^*_{20} + U_{31}U^*_{30})(1) \\
&= 0, \quad (5.2.34)
\end{aligned}$$

where we have used Eq.(5.2.24), Eq.(5.2.25) and Eq.(5.2.33). Next, we can calculate $(\mu_0|\mu_0)$.

Since $\langle u_0|u_0\rangle = 1$, we have

$$U_{00}^*U_{00} + U_{10}^*U_{10} + U_{20}^*U_{20} + U_{30}^*U_{30} = 1, \tag{5.2.35}$$

then

$$\begin{aligned}
(\mu_0|\mu_0) &= \mathrm{tr}(|\mu_0)(\mu_0|) \\
&= \mathrm{tr}([U_{00}|0) + U_{10}|1) + U_{20}|2) + U_{30}|3)][U_{00}^*(0| + U_{10}^*(1| + U_{20}^*(2| + U_{30}^*(3|]) \\
&= [U_{00}U_{00}^*\mathrm{tr}(|0)(0|) + U_{00}U_{10}^*\mathrm{tr}(|0)(1|) + U_{00}U_{20}^*\mathrm{tr}(|0)(2|) + U_{00}U_{30}^*\mathrm{tr}(|0)(3|)] + \\
&\quad [U_{10}U_{00}^*\mathrm{tr}(|1)(0|) + U_{10}U_{10}^*\mathrm{tr}(|1)(1|) + U_{10}U_{20}^*\mathrm{tr}(|1)(2|) + U_{10}U_{30}^*\mathrm{tr}(|1)(3|)] + \\
&\quad [U_{20}U_{00}^*\mathrm{tr}(|2)(0|) + U_{20}U_{10}^*\mathrm{tr}(|2)(1|) + U_{20}U_{20}^*\mathrm{tr}(|2)(2|) + U_{20}U_{30}^*\mathrm{tr}(|2)(3|)] + \\
&\quad [U_{30}U_{00}^*\mathrm{tr}(|3)(0|) + U_{30}U_{10}^*\mathrm{tr}(|3)(1|) + U_{30}U_{20}^*\mathrm{tr}(|3)(2|) + U_{30}U_{30}^*\mathrm{tr}(|3)(3|)] \\
&= (U_{00}U_{00}^* + U_{10}U_{10}^* + U_{20}U_{20}^* + U_{30}U_{30}^*)(2) \\
&= 2,
\end{aligned} \tag{5.2.36}$$

The Schmidt decomposition of a vector can be computed using the source code from QETLAB. We input Eq.(5.2.27) and the source code outputs the matrices, U, D and V. If we see that D has only one non-vanishing element then H_{AC} can be written in a product form.

In order to find out whether $\tilde{H}_{AC}$ can be written as a product $\tilde{H}_A \otimes \tilde{H}_C$, we input $h_{11} = h_{22} = h_{33} = \pi/4$ (and the rest of the entries are all zero) in vector form like in Eq.(5.2.27) and we compute its Schmidt decomposition using MATLAB. It turns out that D has three non-vanishing diagonal entries, each being $\pi/4$. This coincides with Eq.(5.2.21), therefore it is the Schmidt form and there is no way to write it as a product. This completes the proof. □

5.3 Initial state: $|\psi_0\rangle = |\alpha\beta\gamma\rangle$

In this Section, we consider initial states that are in product form as shown above. In Theorem 4, we use Theorem 2 to rewrite H_{AC} and H_{BC} as the subclass of Hamiltonians $\mathcal{H}_1$, and we prove that

$$E_{A:BC}(t) \leq E_{AB:C}(t),$$
$$E_{B:AC}(t) \leq E_{AB:C}(t).$$

Lemma 3 will be useful in proving Theorem 4.

Lemma 3. *For any separable state*

$$\rho = \sum_i p_i \sigma_i^A \otimes \sigma_i^B, \tag{5.3.1}$$

we have $S_{A|B} \geq 0$ *and* $S_{B|A} \geq 0$.

Proof. For any separable state $\rho = \sum_i p_i \sigma_i^A \otimes \sigma_i^B$, we can always introduce ρ' where

$$\rho' = \sum_i p_i \sigma_i^A \otimes \sigma_i^B \otimes |i\rangle_C \langle i|, \tag{5.3.2}$$

such that $\rho = \mathrm{tr}_C(\rho')$.

From strong subadditivity we have, $S_{ABC} + S_B \leq S_{AB} + S_{BC}$, therefore

$$
\begin{aligned}
S_{A|B} &= S_{AB} - S_B \\
&\quad | \text{ Using the definition of strong subadditivity.} \\
&\geq S_{ABC} - S_{BC} \\
&= (S_{ABC} - S_C) - (S_{BC} - S_C) \\
&= S_{AB|C} - S_{B|C} \\
&\quad | \text{ Using the joint entropy theorem.} \\
&= \sum_i p_i S(\sigma_i^A \otimes \sigma_i^B) - \sum_i p_i S(\sigma_i^B) \\
&\quad | \text{ Using the entropy of a tensor product.} \\
&= \sum_i p_i (S(\sigma_i^A) + S(\sigma_i^B)) - \sum_i p_i S(\sigma_i^B) \\
&= \sum_i p_i S(\sigma_i^A) \\
&\geq 0.
\end{aligned} \tag{5.3.3}
$$

Likewise from strong subadditivity, $S_{ABC} + S_A \leq S_{AC} + S_{AB}$,

$$
\begin{aligned}
S_{B|A} &= S_{AB} - S_A \\
&\quad | \text{ Using the definition of strong subadditivity.} \\
&\geq S_{ABC} - S_{AC} \\
&= (S_{ABC} - S_C) - (S_{AC} - S_C) \\
&= S_{AB|C} - S_{A|C} \\
&\quad | \text{ Using the joint entropy theorem.} \\
&= \sum_i p_i S(\sigma_i^A \otimes \sigma_i^B) - \sum_i p_i S(\sigma_i^A) \\
&\quad | \text{ Using the entropy of a tensor product.} \\
&= \sum_i p_i (S(\sigma_i^A) + S(\sigma_i^B)) - \sum_i p_i S(\sigma_i^A) \\
&= \sum_i p_i S(\sigma_i^B) \\
&\geq 0.
\end{aligned} \tag{5.3.4}
$$

□

Theorem 4. *For a subclass of interaction Hamiltonians, $\mathcal{H}_1$, that reads,*

$$H_{AC} = H_A + \tilde{H}_A \otimes \tilde{H}_C, \tag{5.3.5}$$

$$H_{BC} = H_B + \tilde{H}_B \otimes \tilde{H}_C, \tag{5.3.6}$$

where $[H_{AC}, H_{BC}] = 0$ and the total Hamiltonian is $H = H_{AC} + H_{BC}$, if the initial tripartite state is $|\alpha\beta\gamma\rangle$ then for all times,

$$S_A(t) \leq S_C(t), \tag{5.3.7}$$

$$S_B(t) \leq S_C(t). \tag{5.3.8}$$

Proof. Applying Theorem 1, the total Hamiltonian can be rewritten as

$$H = \underbrace{H_A + \tilde{H}_A \otimes \tilde{H}_C}_{H_{AC}} + \underbrace{H_B + \tilde{H}_B \otimes \tilde{H}_C}_{H_{BC}}, \tag{5.3.9}$$

and there is some common eigenbasis $\{|c\rangle\}$ on $\tilde{H}_C$, where $\tilde{H}_C |c\rangle = \tilde{E}_c |c\rangle$. Expressing $|\gamma\rangle$ in this eigenbasis,

$$|\gamma\rangle = \sum_c \gamma_c |c\rangle\,, \tag{5.3.10}$$

where $\sum_c |\gamma_c|^2 = 1$. The initial state reads

$$|\alpha\beta\gamma\rangle = \sum_c \gamma_c |\alpha\beta c\rangle\,. \tag{5.3.11}$$

Since $[H_{AC}, H_{BC}] = 0$, by the Baker-Campbell-Hausdorff formula, the time evolution can be split into

$$e^{-\frac{it}{\hbar}(H_{AC}+H_{BC})} |\alpha\beta c\rangle = \underbrace{e^{-\frac{it}{\hbar}(H_A+\tilde{H}_A\otimes\tilde{H}_C)}}_{①} \underbrace{e^{-\frac{it}{\hbar}(H_B+\tilde{H}_B\otimes\tilde{H}_C)}}_{②} |\alpha\beta c\rangle\,, \tag{5.3.12}$$

where we look at how each $|\alpha\beta c\rangle$ evolves in time.

One can calculate $\left[H_A, \tilde{H}_A \otimes \tilde{H}_C\right]$ and $\left[H_B, \tilde{H}_B \otimes \tilde{H}_C\right]$ and find that they are non-vanishing. This means we cannot split ① and ② further. Here onwards we use the Trotter expansion.

The Trotter expansion allows us to break matrix exponentials into infinitesimal interactions that get repeated infinitely many times. Applying the Trotter expansion to ① and ②, we have

$$e^{-\frac{it}{\hbar}(H_{AC}+H_{BC})}\left|\alpha\beta c\right\rangle = \lim_{n\to\infty}\left[\left(e^{-\frac{i\Delta t}{\hbar}H_A}e^{-\frac{i\Delta t}{\hbar}\tilde{H}_A\otimes\tilde{H}_C}\right)^n\left(e^{-\frac{i\Delta t}{\hbar}H_B}e^{-\frac{i\Delta t}{\hbar}\tilde{H}_B\otimes\tilde{H}_C}\right)^n\right]\left|\alpha\beta c\right\rangle, \quad (5.3.13)$$

where $\Delta t = t/n$. From right to left, applying $e^{-\frac{i\Delta t}{\hbar}\tilde{H}_B\otimes\tilde{H}_C}$ once gives,

$$\begin{aligned} e^{-\frac{i\Delta t}{\hbar}\tilde{H}_B\otimes\tilde{H}_C}\left|\alpha\beta c\right\rangle &= \sum_{m=0}^{\infty}\left(\frac{1}{m!}\right)\left(-\frac{i\Delta t}{\hbar}\right)^m\left(\tilde{H}_B\right)^m\otimes\left(\tilde{H}_C\right)^m\left|\alpha\beta c\right\rangle \\ &= \sum_{m=0}^{\infty}\left(\frac{1}{m!}\right)\left(-\frac{i\Delta t}{\hbar}\right)^m\left(\tilde{H}_B\right)^m\left(\tilde{E}_c\right)^m\left|\alpha\beta c\right\rangle \\ &= e^{-\frac{i\Delta t}{\hbar}\tilde{E}_c\tilde{H}_B}\left|\alpha\beta c\right\rangle. \end{aligned} \quad (5.3.14)$$

Applying $e^{-\frac{i\Delta t}{\hbar}H_B}$ next,

$$\begin{aligned} e^{-\frac{i\Delta t}{\hbar}H_B}e^{-\frac{i\Delta t}{\hbar}\tilde{H}_B\otimes\tilde{H}_C}\left|\alpha\beta c\right\rangle &= \left|\alpha\right\rangle\left(e^{-\frac{i\Delta t}{\hbar}H_B}e^{-\frac{i\Delta t}{\hbar}\tilde{E}_c\tilde{H}_B}\left|\beta\right\rangle\right)\left|c\right\rangle \\ &= \left|\alpha\beta_c^{(1)}c\right\rangle, \end{aligned} \quad (5.3.15)$$

where $\left|\beta_c^{(1)}\right\rangle = \left(e^{-\frac{i\Delta t}{\hbar}H_B}e^{-\frac{i\Delta t}{\hbar}\tilde{E}_c\tilde{H}_B}\left|\beta\right\rangle\right)$. The subscript c shows the dependence on energy $\tilde{E}_c$ and the superscript (1) indicates that we have iterated $\left(e^{-\frac{i\Delta t}{\hbar}H_B}e^{-\frac{i\Delta t}{\hbar}\tilde{H}_B\otimes\tilde{H}_C}\right)$ once. Since the eigenstate $\left|c\right\rangle$ is not modified after one iteration, iterating this n times, where $n\to\infty$, gives us

$$\begin{aligned} \lim_{n\to\infty}\left(e^{-\frac{i\Delta t}{\hbar}H_B}e^{-\frac{i\Delta t}{\hbar}\tilde{H}_B\otimes\tilde{H}_C}\right)^n\left|\alpha\beta c\right\rangle &= \lim_{n\to\infty}\left(\left|\alpha\beta_c^{(n)}c\right\rangle\right) \\ &= \left|\alpha\beta_c^{(\infty)}c\right\rangle, \end{aligned} \quad (5.3.16)$$

where the superscript indicates the number of iterations.

Continuing, we apply $e^{-\frac{i\Delta t}{\hbar}\tilde{H}_A\otimes\tilde{H}_C}$ onto the result above and we get

$$\begin{aligned} e^{-\frac{i\Delta t}{\hbar}\tilde{H}_A\otimes\tilde{H}_C}\left|\alpha\beta_c^{(\infty)}c\right\rangle &= \sum_{m=0}^{\infty}\left(\frac{1}{m!}\right)\left(-\frac{i\Delta t}{\hbar}\right)^m\left(\tilde{H}_A\right)^m\otimes\left(\tilde{H}_C\right)^m\left|\alpha\beta_c^{(\infty)}c\right\rangle \\ &= \sum_{m=0}^{\infty}\left(\frac{1}{m!}\right)\left(-\frac{i\Delta t}{\hbar}\right)^m\left(\tilde{H}_A\right)^m\left(\tilde{E}_c\right)^m\left|\alpha\beta_c^{(\infty)}c\right\rangle \\ &= e^{-\frac{i\Delta t}{\hbar}\tilde{E}_c\tilde{H}_A}\left|\alpha\beta_c^{(\infty)}c\right\rangle. \end{aligned} \quad (5.3.17)$$

Applying $e^{-\frac{i\Delta t}{\hbar}H_A}$ next, we have

$$e^{-\frac{i\Delta t}{\hbar}H_A}e^{-\frac{i\Delta t}{\hbar}\tilde{H}_A\otimes\tilde{H}_C}\left|\alpha\beta_c^{(\infty)}c\right\rangle = \left(e^{-\frac{i\Delta t}{\hbar}H_A}e^{-\frac{i\Delta t}{\hbar}\tilde{E}_c\tilde{H}_A}\left|\alpha\right\rangle\right)\left|\beta_c^{(\infty)}\right\rangle\left|c\right\rangle$$
$$= \left|\alpha_c^{(1)}\beta_c^{(\infty)}c\right\rangle, \tag{5.3.18}$$

where $\left|\alpha_c^{(1)}\right\rangle = \left(e^{-\frac{i\Delta t}{\hbar}H_A}e^{-\frac{i\Delta t}{\hbar}\tilde{E}_c\tilde{H}_A}\left|\alpha\right\rangle\right)$.

Again the eigenstate $|c\rangle$ remains unchanged after one iteration, so iterating this n times where $n\to\infty$ gives us

$$\lim_{n\to\infty}\left(e^{-\frac{i\Delta t}{\hbar}H_A}e^{-\frac{i\Delta t}{\hbar}\tilde{H}_A\otimes\tilde{H}_C}\right)^n\left|\alpha\beta_c^{(\infty)}c\right\rangle = \lim_{n\to\infty}\left(\left|\alpha_c^{(n)}\beta_c^{(\infty)}c\right\rangle\right)$$
$$= \left|\alpha_c^{(\infty)}\beta_c^{(\infty)}c\right\rangle. \tag{5.3.19}$$

Hence, the initial tripartite pure state evolves to:

$$|\alpha\beta\gamma\rangle = \sum_c \gamma_c\,|\alpha\beta c\rangle \to \sum_c \gamma_c\left|\alpha_c^{(\infty)}\beta_c^{(\infty)}c\right\rangle = |\psi_t\rangle. \tag{5.3.20}$$

It is now simple to trace out subsystem C and find the following density matrix for subsystem AB:

$$\rho_{AB}(t) = \sum_c \left|\gamma_c\right|^2\left|\alpha_c^{(\infty)}\right\rangle\left\langle\alpha_c^{(\infty)}\right|\otimes\left|\beta_c^{(\infty)}\right\rangle\left\langle\beta_c^{(\infty)}\right|. \tag{5.3.21}$$

Recalling Section 4.2 on the separability of mixed states, Eq.(5.3.21) shows that ρ_{AB} is separable at all times. From Lemma 3, we have $S_{A|B}\geq 0$ and $S_{B|A}\geq 0$. Finally,

$$\begin{aligned}0 &= S_{AB}(t) - S_C(t)\\ &= S_{B|A}(t) + S_A(t) - S_C(t)\\ &\geq S_A(t) - S_C(t),\end{aligned} \tag{5.3.22}$$

$$\begin{aligned}0 &= S_{AB}(t) - S_C(t)\\ &= S_{A|B}(t) + S_B(t) - S_C(t)\\ &\geq S_B(t) - S_C(t).\end{aligned} \tag{5.3.23}$$

Rearranging, we obtain $S_A(t)\leq S_C(t)$ and $S_B(t)\leq S_C(t)$, which completes the proof. □

Since the relative entropy of entanglement of a bipartite pure state, $E_{X:Y}$, is equivalent to the von Neumann entropy of either subsystem, S_X or S_Y, we can interpret our result as

$$E_{A:BC}(t) \leq E_{AB:C}(t), \tag{5.3.24}$$

$$E_{AC:B}(t) \leq E_{AB:C}(t). \tag{5.3.25}$$

This is a very intuitive statement. Since our total Hamiltonian is $H_{AC} + H_{BC}$, i.e. subsystems A and B do not interact directly but only via mediator C, one might expect that entanglement between $A(B)$ and other subsystems is bounded by how much entanglement is present in the mediator.

Using MATLAB, we generated 100 commuting Hamiltonians of the form $H_{AC} = H_A + \tilde{H}_{AC}$ and $H_{BC} = H_B + \tilde{H}_{BC}$ and 10,000 random pure product states for each pair of Hamiltonians. Indeed we observe that $S_A(t) \leq S_C(t)$ and $S_B(t) \leq S_C(t)$. Figure 5.3 below shows an exemplary dynamics of a random pure product state evolved using those Hamiltonians.

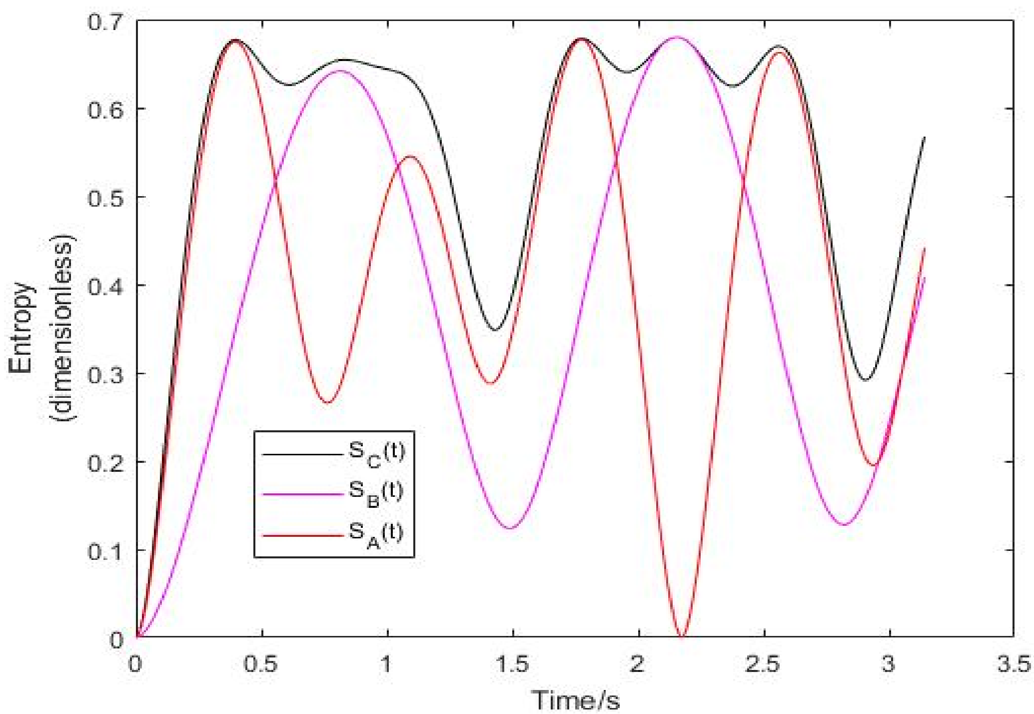

Figure 5.3: A random $|\alpha\beta\gamma\rangle$ initial state evolved using random commuting general Hamiltonians that can be written in the form $H_{AC} = H_A + \tilde{H}_{AC}$ and $H_{BC} = H_B + \tilde{H}_{BC}$.

5.4 Initial state: $|\psi_0\rangle = |\chi\rangle_{AB} |\gamma\rangle_C$

In this Section, we look at bi-product states $|\chi\rangle_{AB} |\gamma\rangle_C$, where A and B are initially entangled. We sampled 100 commuting interaction Hamiltonians of the form $H_{AC} = H_A + \tilde{H}_{AC}$ and $H_{BC} = H_B + \tilde{H}_{BC}$ for 10,000 random bi-product states each. Our numerical results suggests

$$\begin{aligned} &S_A(t) - S_A(0) \leq S_C(t), \\ &S_B(t) - S_B(0) \leq S_C(t). \end{aligned} \tag{5.4.1}$$

This is rather intuitive since it reduces to the bound we proved in Theorem 4 for pure product states ($S_A(0) = S_B(0) = 0$ for $|\alpha\beta\gamma\rangle$). One can interpret this as the entanglement distributed to A/B has to be less than the entanglement communicated by C. Although we are unable to prove this bound analytically, below we present a plot that illustrates the tightness of this bound.

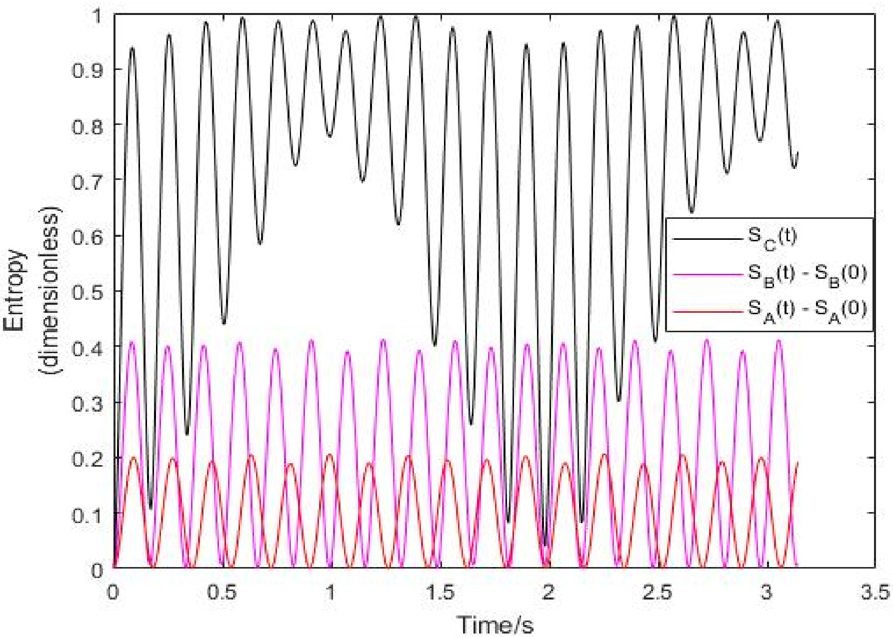

Figure 5.4: A random $|\chi\rangle_{AB} |\gamma\rangle_C$ initial state evolved using random commuting general Hamiltonians that can be written in the form $H_{AC} = H_A + \tilde{H}_A \otimes \tilde{H}_C$ and $H_{BC} = H_B + \tilde{H}_B \otimes \tilde{H}_C$. The black line is $S_C(t)$, the purple line is $S_A(t) - S_A(0)$ and the red line is $S_B(t) - S_B(0)$.

In a bid to prove this bound analytically, we have looked at other approaches: trying to simplify the problem or restrict ourselves to specific Hamiltonians or specific initial states. Amongst these approaches, we managed to prove $|S_A(t) - S_A(0)| \leq S_C(t)$ for the case where H_{AC} realizes the swap operation S_{A-C} at some particular time and H_{BC} is the identity operator. We present the proof of this in Theorem 5.

In Theorem 6, we continue to look at bi-product states $|\chi\rangle_{AB}|\gamma\rangle_C$ for commuting interaction Hamiltonians of the form $H_{AC} = H_A + \tilde{H}_A \otimes \tilde{H}_C$ and $H_{BC} = H_B + \tilde{H}_B \otimes \tilde{H}_C$. We show that unitary dynamics comes into play and prove that $S_A(t) \geq S_A(0)$ and $S_B(t) \geq S_B(0)$. Using Theorem 6, we find $S_A(0) - S_A(t) \leq S_C(t)$ and $S_B(0) - S_B(t) \leq S_C(t)$ which is analogous to part of what we had in Theorem 5.

Theorem 5. *For an initial state* $|\psi_0\rangle = |\chi\rangle_{AB}|\gamma\rangle_C$ *and interaction Hamiltonians of the form*

$$H_{AC} = -\frac{\pi}{4}\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1} + \frac{\pi}{4}\sigma_1\otimes\mathbb{1}\otimes\sigma_1 + \frac{\pi}{4}\sigma_2\otimes\mathbb{1}\otimes\sigma_2 + \frac{\pi}{4}\sigma_3\otimes\mathbb{1}\otimes\sigma_3, \tag{5.4.2}$$

$$H_{BC} = \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}, \tag{5.4.3}$$

where $[H_{AC}, H_{BC}] = 0$ *the total Hamiltonian is* $H = H_{AC} + H_{BC}$*, we have at any time:*

$$|S_A(t) - S_A(0)| \leq S_C(t). \tag{5.4.4}$$

Proof. Since $[H_{AC}, H_{BC}] = 0$, by the Baker-Campbell-Hausdorff formula, the time evolution of our initial state reads

$$|\psi_t\rangle = e^{-\frac{it}{\hbar}(H_{AC}+H_{BC})}|\psi_0\rangle = e^{-\frac{it}{\hbar}H_{AC}}e^{-\frac{it}{\hbar}H_{BC}}|\psi_0\rangle. \tag{5.4.5}$$

Since H_{BC} is the identity, it just adds a global phase of $e^{-\frac{it}{\hbar}}$. Recalling that $U_{AC} = e^{-\frac{it}{\hbar}H_{AC}}$ from Eq.(5.2.14),

$$|\psi_t\rangle = e^{-\frac{it}{\hbar}}U_{AC}|\psi_0\rangle. \tag{5.4.6}$$

$\rho_{ABC}(t)$ then reads

$$\rho_{ABC}(t) = |\psi_t\rangle\langle\psi_t| = U_{AC}\rho_{ABC}(0)U_{AC}^\dagger. \tag{5.4.7}$$

Since unitary evolution does not change the entropy, $S_{AC}(t) = S_{AC}(0)$. Since the evolved state $|\psi_t\rangle$ and the initial state $|\psi_0\rangle$ are pure, we have $S_{AC}(t) = S_B(t)$ and $S_{AC}(0) = S_B(0)$. Combining, we get $S_B(t) = S_B(0)$. This is intuitive since our interaction Hamiltonians do not change the state of B.

Since the mutual information between any two parties is always non-negative, we can write

$$S_A(t) + S_C(t) - S_{AC}(t) \geq 0 \tag{5.4.8}$$

Using $S_{AC}(t) = S_{AC}(0) = S_B(0) = S_A(0)$, we get

$$\begin{aligned} S_A(t) - S_{AC}(0) &\geq -S_C(t) \\ S_A(t) - S_A(0) &\geq -S_C(t) \\ S_A(0) - S_A(t) &\leq S_C(t), \end{aligned} \tag{5.4.9}$$

where $S_B(0) = S_A(0)$ because $|\chi\rangle_{AB}$ is pure. Using strong subadditivity, we find

$$\begin{aligned} S_{ABC}(t) + S_A(t) &\leq S_{AC}(t) + S_{AB}(t) \\ 0 + S_A(t) &\leq S_{AC}(t) + S_{AB}(t) \\ &= S_A(0) + S_C(t). \end{aligned} \tag{5.4.10}$$

Combining Eq.(5.4.9) and Eq.(5.4.10), we get

$$|S_A(t) - S_A(0)| \leq S_C(t). \tag{5.4.11}$$

This completes the proof. □

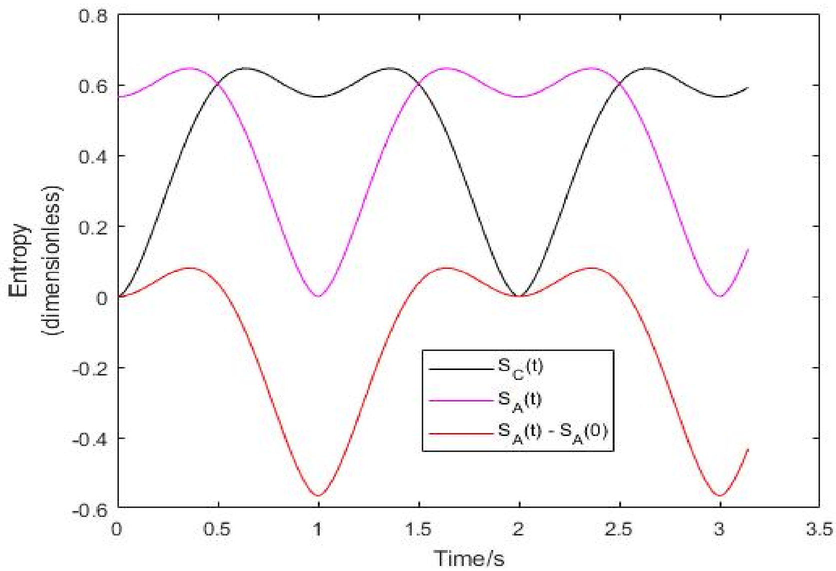

Figure 5.5: A random $|\chi\rangle_{AB}\,|\gamma\rangle_C$ initial state evolved using H_{AC} that generates the swap S_{A-C} at $t = 1$ and H_{BC} = Identity. We set $\hbar = \omega = 1$.

If we look at Figure 5.5 (we set $\hbar = \omega = 1$), from $0 \leq t \leq 2$, we see it is symmetrical about $t = 1$. At $t = 1$, the swap is completed. A and C have successfully exchanged their initial states. From $1 \leq t \leq 2$, the graph is a reflection of itself from $0 \leq t \leq 1$ since it is swapping back to what we had originally. We also see that $S_A(t) = S_C(t+1)$. This can be shown easily. Using the Schmidt decomposition, the initial state reads,

$$|\psi_0\rangle = \sqrt{p}\,|a\rangle_A\,|b\rangle_B\,|\gamma\rangle_C + \sqrt{1-p}\,|a_\perp\rangle_A\,|b_\perp\rangle_B\,|\gamma\rangle_C\,. \tag{5.4.12}$$

Using Eq.(5.4.6) and U_{AC} from Eq.(5.2.18), the state at time t reads

$$\begin{aligned}|\psi_t\rangle = e^{-\frac{it}{\hbar}}(e^{i\pi\omega t}+1)(1/2)(\sqrt{p}\,|a\rangle\,|b\rangle\,|\gamma\rangle + \sqrt{1-p}\,|a_\perp\rangle\,|b_\perp\rangle\,|\gamma\rangle) +\\ e^{-\frac{it}{\hbar}}(1-e^{-i\pi\omega t})(1/2)(\sqrt{p}\,|\gamma\rangle\,|b\rangle\,|a\rangle + \sqrt{1-p}\,|\gamma\rangle\,|b_\perp\rangle\,|a_\perp\rangle).\end{aligned} \tag{5.4.13}$$

The density operator of the composite system at time t is just $\rho_t = |\psi_t\rangle\langle\psi_t|$. Setting $\omega = 1$ and pushing through the math, the reduced density operators of subsystem A and C read

$$\begin{aligned}\rho_A(t) = {}&(1/2)(1+\cos(\pi t))(p\,|a\rangle\langle a| + (1-p)\,|a_\perp\rangle\langle a_\perp|) + (1/2)(1-\cos(\pi t))\,|\gamma\rangle\langle\gamma| \;+\\ &(1/4)(1-e^{i2\pi t})(p\,\langle a|\gamma\rangle\,|a\rangle\langle\gamma| + (1-p)\,\langle a_\perp|\gamma\rangle\,|a_\perp\rangle\langle\gamma|) +\\ &(1/4)(1-e^{-i2\pi t})(p\,\langle\gamma|a\rangle\,|\gamma\rangle\langle a| + (1-p)\,\langle\gamma|a_\perp\rangle\,|\gamma\rangle\langle a_\perp|),\end{aligned} \tag{5.4.14}$$

$$\begin{aligned}\rho_C(t) = {}&(1/2)(1+\cos(\pi t))\,|\gamma\rangle\langle\gamma| + (1/2)(1-\cos(\pi t))(p\,|a\rangle\langle a| + (1-p)\,|a_\perp\rangle\langle a_\perp|) +\\ &(1/4)(1-e^{i2\pi t})(p\,\langle\gamma|a\rangle\,|\gamma\rangle\langle a| + (1-p)\,\langle\gamma|a_\perp\rangle\,|\gamma\rangle\langle a_\perp|) +\\ &(1/4)(1-e^{-i2\pi t})(p\,\langle a|\gamma\rangle\,|a\rangle\langle\gamma| + (1-p)\,\langle a_\perp|\gamma\rangle\,|a_\perp\rangle\langle\gamma|).\end{aligned} \tag{5.4.15}$$

Substituting $\cos(\pi(t+1)) = \cos(\pi t + \pi) = \cos(\pi t)\cos(\pi) - \sin(\pi t)\sin(\pi) = -\cos(\pi t)$ into Eq.(5.4.15), and since $e^{\pm i2\pi(t+1)} = e^{\pm i2\pi t}$, one easily finds that $\rho_A(t) = \rho_C(t+1)$. Therefore it is not surprising that $S_A(t)$ and $S_C(t)$ are translations of each other by a period of 1. In general if we had not set $\omega = 1$, we would observe

$$S_A(t) = S_C(t + 1/\omega). \tag{5.4.16}$$

When A and B are initially minimally entangled, $S_A(t) - S_A(0) \leq S_C(t)$ forms a tight bound. In Figure 5.6, we give an example where we consider the initial state $|\psi_0\rangle = (\sqrt{0.99}\,|01\rangle + \sqrt{0.01}\,|10\rangle)\,|1\rangle$.

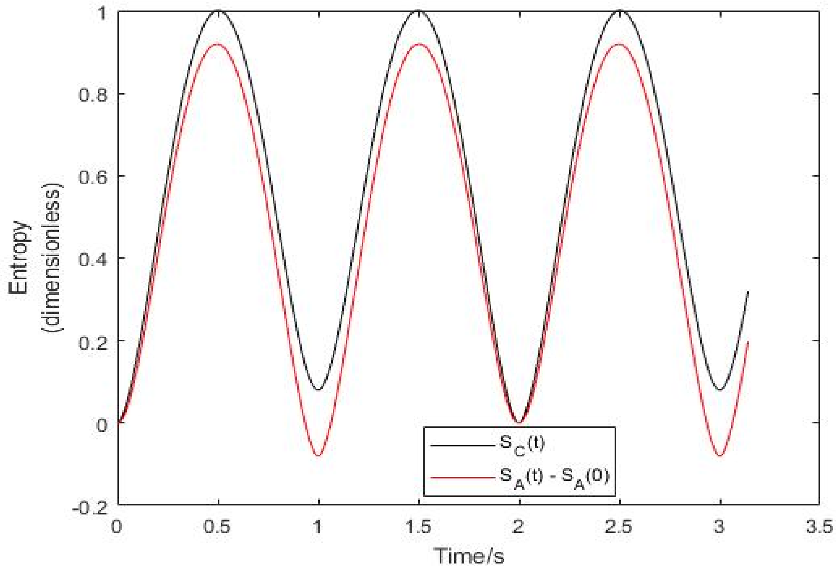

Figure 5.6: Illustrating a tight bound for the initial state $|\psi_0\rangle = (\sqrt{0.99}\,|01\rangle + \sqrt{0.01}\,|10\rangle)\,|1\rangle$.

Also, one might expect the equality condition happens when A and B are not entangled initially (from looking at Figure 5.6). We can prove this using strong subadditivity,

$$S_{ABC}(t) + S_C(t) \leq S_{BC}(t) + S_{AC}(t). \tag{5.4.17}$$

Since the evolved tripartite state is pure, $S_{ABC}(t) = 0$, $S_{BC}(t) = S_A(t)$ and $S_{AC}(t) = S_B(t)$. The state of B is unchanged throughout our evolution and A and B are initially in product form, so $S_B(t) = S_B(0) = 0$. Eq.(5.4.17) reads,

$$\begin{aligned} 0 + S_C(t) &\leq S_A(t) + S_B(t) \\ &= S_A(t) + S_B(0) \\ &= S_A(t) + 0. \end{aligned} \tag{5.4.18}$$

Rearranging, we get $S_A(t) \geq S_C(t)$, and from Theorem 5 we have $S_A(t) \leq S_C(t)$. Therefore the only way for this to make sense is for $S_A(t) = S_C(t)$.

Theorem 6. *For a subclass of interaction Hamiltonians, $\mathcal{H}_1$, that reads,*

$$H_{AC} = H_A + \tilde{H}_A \otimes \tilde{H}_C, \tag{5.4.19}$$

$$H_{BC} = H_B + \tilde{H}_B \otimes \tilde{H}_C, \tag{5.4.20}$$

and $[H_{AC}, H_{BC}] = 0$, if the initial tripartite state is $|\chi\rangle_{AB} |\gamma\rangle_C$, then

$$S_A(t) \geq S_A(0), \tag{5.4.21}$$

$$S_B(t) \geq S_B(0). \tag{5.4.22}$$

Proof. Let us define the eigendecomposition of $\tilde{H}_C$ as

$$\tilde{H}_C = c |c\rangle \langle c| + c_\perp |c_\perp\rangle \langle c_\perp| , \tag{5.4.23}$$

where c and $c_\perp$ are energy eigenvalues and $\{|c\rangle, |c_\perp\rangle\}$ are the eigenstates of $\tilde{H}_C$. Expressing $|\gamma\rangle$ as eigenstates of $\tilde{H}_C$ and applying the Schmidt decomposition to AB, our initial state reads,

$$|\psi_0\rangle = (\sqrt{p} |a\rangle |b\rangle + \sqrt{1-p} |a_\perp\rangle |b_\perp\rangle)(\gamma_1 |c\rangle + \gamma_2 |c_\perp\rangle), \tag{5.4.24}$$

where $|\gamma_1|^2 + |\gamma_2|^2 = 1$ and p is probability. Since $[H_{AC}, H_{BC}] = 0$, by the Baker-Campbell-Hausdorff formula, the time evolution of our initial state reads,

$$\begin{aligned}
|\psi_t\rangle &= e^{-\frac{it}{\hbar}(H_{AC}+H_{BC})} |\psi_0\rangle \\
&= e^{-\frac{it}{\hbar}(H_{AC})} e^{-\frac{it}{\hbar}(H_{BC})} |\psi_0\rangle \\
&= e^{-\frac{it}{\hbar}(H_A+\tilde{H}_A\otimes\tilde{H}_C)} e^{-\frac{it}{\hbar}(H_B+\tilde{H}_B\otimes\tilde{H}_C)} |\psi_0\rangle .
\end{aligned} \tag{5.4.25}$$

Applying the Trotter expansion, we find

$$\begin{aligned}
|\psi_t\rangle &= \lim_{n\to\infty} \left[\left(e^{-\frac{i\Delta t}{\hbar} H_A} e^{-\frac{i\Delta t}{\hbar} \tilde{H}_A \otimes \tilde{H}_C} \right)^n \left(e^{-\frac{i\Delta t}{\hbar} H_B} e^{-\frac{i\Delta t}{\hbar} \tilde{H}_B \otimes \tilde{H}_C} \right)^n \right] |\psi_0\rangle \\
&\quad | \quad \text{The steps are the same as what we did in Theorem 4.} \\
&\quad | \quad \text{We will skip straight to the result. One may refer to Theorem 4 if need be.} \\
&= (\sqrt{p} |a_c^\infty\rangle |b_c^\infty\rangle + \sqrt{1-p} |a_c^\infty \perp\rangle |b_c^\infty \perp\rangle \gamma_1 |c\rangle \ + \\
&\quad (\sqrt{p} |a_{c_\perp}^\infty\rangle |b_{c_\perp}^\infty\rangle + \sqrt{1-p} |a_{c_\perp}^\infty \perp\rangle |b_{c_\perp}^\infty \perp\rangle \gamma_2 |c_\perp\rangle ,
\end{aligned} \tag{5.4.26}$$

where

$$
\begin{aligned}
|a_c^\infty\rangle &= (e^{-\frac{i\Delta t}{\hbar}H_A}e^{-\frac{i\Delta t}{\hbar}c\tilde{H}_A})^\infty |a\rangle\,, \quad |b_c^\infty\rangle = (e^{-\frac{i\Delta t}{\hbar}H_B}e^{-\frac{i\Delta t}{\hbar}c\tilde{H}_B})^\infty |b\rangle \\
|a_c^\infty \perp\rangle &= (e^{-\frac{i\Delta t}{\hbar}H_A}e^{-\frac{i\Delta t}{\hbar}c\tilde{H}_A})^\infty |a_\perp\rangle\,, \quad |b_c^\infty \perp\rangle = (e^{-\frac{i\Delta t}{\hbar}H_B}e^{-\frac{i\Delta t}{\hbar}c\tilde{H}_B})^\infty |b_\perp\rangle \\
\left|a_{c_\perp}^\infty\right\rangle &= (e^{-\frac{i\Delta t}{\hbar}H_A}e^{-\frac{i\Delta t}{\hbar}c_\perp\tilde{H}_A})^\infty |a\rangle\,, \quad \left|b_{c_\perp}^\infty\right\rangle = (e^{-\frac{i\Delta t}{\hbar}H_B}e^{-\frac{i\Delta t}{\hbar}c_\perp\tilde{H}_B})^\infty |b\rangle \\
\left|a_{c_\perp}^\infty \perp\right\rangle &= (e^{-\frac{i\Delta t}{\hbar}H_A}e^{-\frac{i\Delta t}{\hbar}c_\perp\tilde{H}_A})^\infty |a_\perp\rangle\,, \quad \left|b_{c_\perp}^\infty \perp\right\rangle = (e^{-\frac{i\Delta t}{\hbar}H_B}e^{-\frac{i\Delta t}{\hbar}c_\perp\tilde{H}_B})^\infty |b_\perp\rangle
\end{aligned} \tag{5.4.27}
$$

One can show that $|a_c^\infty\rangle$ and $|a_c^\infty \perp\rangle$ are orthogonal (and the same applies for $\left\{\left|a_{c_\perp}^\infty\right\rangle, \left|a_{c_\perp}^\infty \perp\right\rangle\right\}$, $\{|b_c^\infty\rangle, |b_c^\infty \perp\rangle\}$ and $\left\{\left|b_{c_\perp}^\infty\right\rangle, \left|b_{c_\perp}^\infty \perp\right\rangle\right\}$). In the process, we will show that $(e^{-\frac{i\Delta t}{\hbar}H_A}e^{-\frac{i\Delta t}{\hbar}c\tilde{H}_A})^\infty$ is unitary and it preserves the orthogonality between $|a\rangle$ and $|a_\perp\rangle$ (and the rest will follow).

Consider one iteration. We have,

$$
\left|a_c^1\right\rangle = (e^{-\frac{i\Delta t}{\hbar}H_A}e^{-\frac{i\Delta t}{\hbar}c\tilde{H}_A})\,|a\rangle\,, \tag{5.4.28}
$$

$$
\left|a_c^1 \perp\right\rangle = (e^{-\frac{i\Delta t}{\hbar}H_A}e^{-\frac{i\Delta t}{\hbar}c\tilde{H}_A})\,|a_\perp\rangle\,. \tag{5.4.29}
$$

Then,

$$
\begin{aligned}
\left\langle a_c^1 \middle| a_c^1 \perp\right\rangle &= \langle a|\, e^{\frac{i\Delta t}{\hbar}c\tilde{H}_A}(e^{\frac{i\Delta t}{\hbar}H_A}e^{-\frac{i\Delta t}{\hbar}H_A})e^{-\frac{i\Delta t}{\hbar}c\tilde{H}_A}\,|a_\perp\rangle \\
&= \langle a|\, e^{\frac{i\Delta t}{\hbar}c\tilde{H}_A}(\mathbb{1})e^{-\frac{i\Delta t}{\hbar}c\tilde{H}_A}\,|a_\perp\rangle \\
&= \langle a|\,\mathbb{1}\,|a_\perp\rangle \\
&= 0.
\end{aligned} \tag{5.4.30}
$$

One can see that if this is the case for one iteration, this will still be the case for infinitely many iterations. Since they preserve orthogonality, we can treat them as unitaries. Putting this aside first, we find the reduced density matrix on A to be

$$
\begin{aligned}
\rho_A(t) &= \mathrm{tr}_{BC}(|\psi_t\rangle\langle\psi_t|) \\
&= |\gamma_1|^2(p\,|a_c^\infty\rangle\langle a_c^\infty| + (1-p)\,|a_c^\infty \perp\rangle\langle a_c^\infty \perp|) + |\gamma_2|^2(p\left|a_{c_\perp}^\infty\right\rangle\left\langle a_{c_\perp}^\infty\right| + (1-p)\left|a_{c_\perp}^\infty \perp\right\rangle\left\langle a_{c_\perp}^\infty \perp\right|).
\end{aligned} \tag{5.4.31}
$$

Denoting unitaries $U_1 = (e^{-\frac{i\Delta t}{\hbar}H_A}e^{-\frac{i\Delta t}{\hbar}c\tilde{H}_A})^\infty$ and $U_2 = (e^{-\frac{i\Delta t}{\hbar}H_A}e^{-\frac{i\Delta t}{\hbar}c_\perp\tilde{H}_A})^\infty$, we can rewrite Eq.(5.4.31) as,

$$\begin{aligned}\rho_A(t) &= |\gamma_1|^2(pU_1\,|a\rangle\,\langle a|\,U_1^\dagger + (1-p)U_1\,|a_\perp\rangle\,\langle a_\perp|\,U_1^\dagger) + |\gamma_2|^2(pU_2\,|a\rangle\,\langle a|\,U_2^\dagger + (1-p)U_2\,|a_\perp\rangle\,\langle a_\perp|\,U_2^\dagger)\\ &= |\gamma_1|^2U_1(p\,|a\rangle\,\langle a| + (1-p)\,|a_\perp\rangle\,\langle a_\perp|)U_1^\dagger + |\gamma_2|^2U_2(p\,|a\rangle\,\langle a| + (1-p)\,|a_\perp\rangle\,\langle a_\perp|)U_2^\dagger\\ &= |\gamma_1|^2U_1\rho_A(0)U_1^\dagger + |\gamma_2|^2U_2\rho_A(0)U_2^\dagger, \end{aligned} \tag{5.4.32}$$

where we note that $\rho_A(0)$ is just $p\,|a\rangle\,\langle a| + (1-p)\,|a_\perp\rangle\,\langle a_\perp|$.

The concavity of the entropy was stated in [12] as

$$S(\sum_i p_i\rho_i) \geq \sum_i p_iS(\rho_i). \tag{5.4.33}$$

We notice that $\rho_A(t)$ takes the form of $\sum_i p_i\rho_i$, where γ's are probabilities and $U_i\rho_A(0)U_i^\dagger$ form the ρ_i's. $S(\rho_i) = S(\rho_A(0))$ since unitary operations do not change entropy. The above then reads,

$$\begin{aligned} S(\rho_A(t)) &\geq \sum_i p_iS(\rho_A(0))\\ &= S(\rho_A(0))\sum_i p_i\\ &= S(\rho_A(0)). \end{aligned} \tag{5.4.34}$$

The same holds for B. Summing up,

$$\begin{aligned} S_A(t) &\geq S_A(0),\\ S_B(t) &\geq S_B(0). \end{aligned} \tag{5.4.35}$$

This completes the proof. □

Even though we are unable to prove the bound in Eq.(5.4.1) that we had from extensive numerical search, we are able to prove the other way around that $S_A(0) - S_A(t) \leq S_C(t)$ and $S_B(0) - S_B(t) \leq S_C(t)$. The proof goes as follows.

Proof. From strong subadditivity, we have

$$\begin{aligned} S_{ABC}(t) + S_A(t) &\leq S_{AC}(t) + S_{AB}(t), \\ S_A(t) &\leq S_B(t) + S_C(t), \end{aligned} \tag{5.4.36}$$

since the evolved tripartite state $|\psi_t\rangle$ is pure.

Using Theorem 6, and the fact that $|\chi\rangle_{AB}$ is pure, we are able to write,

$$S_B(0) = S_A(0) \leq S_A(t) \leq S_B(t) + S_C(t). \tag{5.4.37}$$

Rearranging, we end up with $S_B(0) - S_B(t) \leq S_C(t)$. The same holds for A. Summing up,

$$\begin{aligned} S_A(0) - S_A(t) &\leq S_C(t), \\ S_B(0) - S_B(t) &\leq S_C(t). \end{aligned} \tag{5.4.38}$$

□

This statement turns out to be trivial since Theorem 6 ensures that the LHS of the inequality is less than or equal to zero. Since the von Neumann entropy is a non-negative quantity, it is natural that $S_C(t)$ will be greater than some value that is less than or equal to zero.

Chapter 6

Conclusion

In this thesis, we started off introducing entanglement distribution between A and B via ancilla C, where A and B do not interact with each other directly but only via C. We looked at what it means for general Hamiltonians H_{AC} and H_{BC} to commute. This led us to two subclasses of Hamiltonians. The subclass of Hamiltonians $\mathcal{H}_1$ corresponds to having A and B continuously interact with C. Then the common eigenbasis will be product on C. The subclass of Hamiltonians $\mathcal{H}_2$ corresponds to isolating either A and B from interacting with C. Then the common eigenbasis would not be product on C. We explicitly considered the case where H_{AC} generates the swap operator S_{A-C} at a particular time and swaps the state of A with C while H_{BC} is just the identity operator. In this case, effectively B does not interact with C. Yet, for initial bi-product states $|\chi\rangle_{AB} |\gamma\rangle_C$, we managed to prove a suggestive bound: $|S_A(t) - S_A(0)| \leq S_C(t)$. For these two classes of Hamiltonians, we considered pure product states $|\psi_0\rangle = |\alpha\beta\gamma\rangle$ and bi-product states $|\psi_0\rangle = |\chi\rangle_{AB} |\gamma\rangle_C$. We showed some non-trivial entropic bounds that shed light on continuous entanglement distribution picture.

6.1 Future directions

We were unable to prove for the subclass of Hamiltonians $\mathcal{H}_1$ and initial states $|\chi\rangle_{AB} |\gamma\rangle_C$ that

$$S_A(t) - S_A(0) \leq S_C(t),$$
$$S_B(t) - S_B(0) \leq S_C(t).$$

We tested 100 pairs of commuting Hamiltonians of that form for 10,000 initial states each and there have been no counterexamples that violate it. The bound seems intuitive and numerical results seem optimistic. A future research direction would be to prove this bound analytically.

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