QUANTUM ENTANGLEMENT

MONOGAMY RELATIONS AND APPLICATIONS

SUBMITTED

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Abstract

Quantum entanglement is arguably at the heart of quantum information and quantum computation. Although the problem of identification of entanglement in pure states has been resolved, alternative criteria for the existence of entanglement are still interesting problems. In this project, such criteria are investigated using monogamy relations. First, a complete set of monogamy relations based on the positive-semidefiniteness of density operators is developed. Next, new monogamy relations as results of correlation complementarity are introduced. A criterion for the existence of entanglement in pure states that involves correlations between all the parties is conjectured. The proofs for systems of two and three qubits are also presented. Finally, implication of monogamy relations on the fidelity of Remote State Preparation is discussed.

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Chapter 1

Introduction

1.1 Background

Quantum mechanics was categorically one of the most successful discoveries of the 20^{th} century. Since its first formulation in the 1900s, quantum mechanics has opened a bizarre full-of-wonder world to not only physicists but also mathematicians, chemists and even biologists. From wave-particle duality to semiconductor, it has proved itself to be an essential field of study for years to come. Quantum information, a branch of quantum mechanics, concerns the information science in presence of quantum effects. Quantum information studies quantum bits, or qubits, and theirs fascinating properties such as quantum entanglement together with their applications in communication, computation and cryptography.

1.1.1 Uncertainty relations

Uncertainty relations are mathematical inequalities that set a limit on the precision at which one may know a certain pair of physical quantities. For example, the more precisely the position of some particle is determined, the less precisely its momentum can be known, and vice versa [1]. In 1927, a formal version of the statement was introduced by Kennard [2],

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}, \tag{1.1}$$

where Δx and Δp_x are the standard deviation of the measurement of x and p_x accordingly.

As quantum formalism progressed, more uncertainty relations were discovered. One of those relates the uncertainty of measurements of Pauli operators, σ_1, σ_2 and σ_3, in any physical state ρ [3]:

$$\Delta^2\sigma_1 + \Delta^2\sigma_2 + \Delta^2\sigma_3 \geq 2, \tag{1.2}$$

where

$$\Delta^2\sigma_j = \left\langle \sigma_j^2 \right\rangle - \left\langle \sigma_j \right\rangle^2 = 1 - \left\langle \sigma_j \right\rangle^2, \quad (j = 1, 2, 3) \tag{1.3}$$

Here $\langle A \rangle$ is the expectation value of operator A. Therefore equation (1.2) can also be written in the form

$$\left\langle \sigma_1 \right\rangle^2 + \left\langle \sigma_2 \right\rangle^2 + \left\langle \sigma_3 \right\rangle^2 \leq 1. \tag{1.4}$$

Although the inequality (1.4) does not explicitly involve any expressions of uncertainty, it is genuinely equivalent to a uncertainty relation. Hence it is natural to extend the scope of uncertainty relations to cover inequalities that involves trade-offs of knowledge one may know about the expectation values of different operators in a physical state. For example, equation (1.4) tells us the expectation values of σ_1, σ_2 and σ_3 can not all be 1, despite that they can individually be. If one wishes to increase the expectation value of σ_1, one needs to sacrifice either σ_2 or σ_3 or both. Inequalities of this kind are also referred to as monogamy relations if they involves more than one system.

The term "monogamy" is often understood in slightly different ways. In quantum entanglement, if system A is maximally entangled with system B, it cannot be entangled with another system C, and vice versa [4, 5, 6]. Therefore, A is said to be monogamous. On the other hand, monogamy relations also refer to the trade-offs between strengths of violations of a Bell inequality [7, 8, 9, 10, 11, 12, 13]. In this project, we are interested in monogamy relations of quantum correlations [7, 14]. They are all direct consequences of the fact that ρ is a physical state, that is the inequalities hold for all states. However, the converse statement is not true. If the inequalities do hold, there are still chances that the state is not physical, i.e ρ is not positive-semidefinite. The problem of finding a complete set of inequalities ensuring that ρ is positive-semidefinite is still open.

1.1.2 Quantum entanglement

The term "Entanglement" was first mentioned in 1935 by Schrödinger in his reply to Einstein-Podolsy-Rosen (EPR) paradox. Since then, it has been the topic of debate that troubled even the greatest minds in physics. As an example, consider the thought experiment in which Alice and Bob shares a pair of daughter particles from a fission reaction. Each of the daughters can have spin up or down. Since the parent nucleus has spin zeros, if one of the daughters has spin up, the other must have spin down. However when given to Alice, her particle has its spin in the mixture of up and down and only after she performs measurements does the spin collapse to either state. The joined state of two particles can be written as

$$\left|\Psi^{-}\right\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B\right), \tag{1.5}$$

such that the qubit A is given to Alice and qubit B belongs to Bob. Then Alice and Bob move light years away from each other and Alice measures her qubit. If the result is $|\uparrow\rangle$, Bob's qubit also collapses to $|\downarrow\rangle$ instantaneously, although they are separated. Einstein argued that there had to be elements of reality hidden in the physical states, and hence quantum mechanics should be incomplete. Not until 1964 when Bell introduced the famous Bell's inequality did the problem settled. The inequality gave an upper bound on the total correlation a system obeying local realism might have. Local realism was one of the fundamental assumptions in Einstein's argument. Subsequent experiments showed that some quantum system actually beat this bound, and hence the observed correlations could not be explained by local realism. Quantum entanglement is a necessary resource to observe the violation.

The question is now whether one could say anything about the nature of entanglement of a system after he performs some local measurements in the laboratory. Fortunately the answer is yes. Non-entangled system satisfies some monogamy relations that entangled systems do not. Thus these inequalities form criteria for the existence of entanglement. Although some of such criteria are well known, finding new monogamy relations that identify entanglement is still an interesting problem.

Quantum bits

Zeros and ones signal, or bits, are the unit of classical information. In the same manner, qubit is the unit of quantum information. Qubit is a quantum two-level system, for example polarization of a photon or z component of an electron's spin. Unlike classical bits which value must be either 0 or 1, qubits can be and usually are in superposition state of 0 and 1. The general state of a qubit is often written as

$$|\Psi\rangle = \alpha |0\rangle + \beta |1\rangle , \tag{1.6}$$

where $|0\rangle , |1\rangle$ are eigenkets of a binary observable like σ_z, and α, β are two complex coefficients. As a rule, $|\Psi\rangle$ is made normalized, that is $\langle\Psi|\Psi\rangle = 1$, by requiring $|\alpha|^2 + |\beta|^2 = 1$. It is often referred to as pure state, in discrimination with mixed states.

Pure states of more than one qubit can be either non-entangled or entangled. Non-entangled states, also called separable or product states, are those that can be written in the form

$$|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle \otimes \ldots |\Psi_N\rangle , \tag{1.7}$$

where $|\Psi_1\rangle , |\Psi_2\rangle , \ldots , |\Psi_N\rangle$ are some pure states of the 1^{st}, 2^{nd},..., N^{th} qubit respectively, and $\otimes$ stands for the tensor product. Meanwhile, entangled state are those that can not be written in the form (1.7), like $|\Psi^+\rangle$ in (1.5).

Density operator

Instead of the wavevector representation $|\Psi\rangle$, one may use the density operator defined as

$$\rho = |\Psi\rangle \langle\Psi| . \tag{1.8}$$

If $|\Psi\rangle$ is product, ρ is also product

$$\rho = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_N, \tag{1.9}$$

and vice versa. In recent studies, density operators are preferred over the wavevectors as they can be generalized with ease to mixed states, those that can not be written as (1.8). For example,

$$\rho = \frac{1}{2} \left(|0\rangle \langle 0| + |1\rangle \langle 1| \right), \tag{1.10}$$

is a mixed state. There are two important properties of density operators. First, they are Hermitian

$$\rho^\dagger = \left(|\Psi\rangle \langle\Psi| \right)^\dagger = |\Psi\rangle \langle\Psi| = \rho. \tag{1.11}$$

And second, all eigenvalues of ρ are non-negative real numbers. The fact that they are real comes from the Hermiticity of density operators. To see that the eigenvalues of density operators are all non-negative, let $\{|j\rangle\}$ be an orthonormal basis in which ρ is diagonal. If one measures the projection of ρ into the $\{|j\rangle\}$, ρ will collapse to one of the states $\{|j\rangle\}$ with the probability

$$\lambda_j = \mathrm{Tr} \left(\rho \; |j\rangle \langle j| \right), \tag{1.12}$$

which is also the j^{th} eigenvalue of ρ. Hence all λ_j must be non-negative. The converse is also true. Every Hermitian matrix which all eigenvalues are real and non-negative is a density operator representing a physical state. This criterion is a crucial point in identifying the complete set of monogamy relations in the later chapter.

Quantum Correlations

Throughout the thesis, the measure of quantum correlations of an N-qubit state will be defined as

$$T_{\mu_1 \ldots \mu_N} = \left\langle \sigma^{(1)}_{\mu_1} \otimes \ldots \otimes \sigma^{(N)}_{\mu_N} \right\rangle_\rho = \mathrm{Tr}(\rho\, \sigma^{(1)}_{\mu_1} \otimes \ldots \otimes \sigma^{(N)}_{\mu_N}), \tag{1.13}$$

where ρ is the density operator as usual, $\sigma^{(k)}_{\mu_k}$ are the Identity ($\mu_k = 0$) and Pauli matrices ($\mu_k = 1, 2, 3$) acting in the space of the k^{th} qubit. $T_{\mu_1 \ldots \mu_N}$ can also be interpreted as the expectation value of $\sigma^{(1)}_{\mu_1} \otimes \ldots \otimes \sigma^{(N)}_{\mu_N}$ in ρ.
The elements $T_{\mu_1 \ldots \mu_N}$ from a tensor T called correlation tensor, or sometimes correlation matrix. For a system of a single qubit, T reduces to a vector known as the Bloch vector. The definition arises as a consequence of the fact that tensor products of Pauli matrices $\sigma^{(1)}_{\mu_1} \otimes \ldots \otimes \sigma^{(N)}_{\mu_N}$ form a basis of the Hilbert space of the multipartite state. It, hence, allows the density operator ρ to be decomposed as

$$\rho = \frac{1}{2^N} \sum_{\mu_1=0}^{3} \ldots \sum_{\mu_N=0}^{3} T_{\mu_1 \ldots \mu_N} \sigma^{(1)}_{\mu_1} \otimes \ldots \otimes \sigma^{(N)}_{\mu_N}. \tag{1.14}$$

Since ρ is normalized and the Pauli matrices are traceless, the following result is straightforward

$$T_{00\ldots0} = 1. \tag{1.15}$$

An important bound on the correlation is obtained immediately from the purity condition

$$\mathrm{Tr}(\rho^2) \leq 1. \tag{1.16}$$

Applying the decomposition (1.14) to the LHS gives

$$\mathrm{Tr}(\rho^2) = \mathrm{Tr}\left(\frac{1}{2^{2N}} \sum_{\mu_1=0}^{3} \ldots \sum_{\mu_N=0}^{3} \sum_{\nu_1=0}^{3} \ldots \sum_{\nu_N=0}^{3} T_{\mu_1 \ldots \mu_N} T_{\nu_1 \ldots \nu_N} \sigma^{(1)}_{\mu_1} \sigma^{(1)}_{\nu_1} \otimes \ldots \otimes \sigma^{(N)}_{\mu_N} \sigma^{(N)}_{\nu_N} \right)$$

$$
\begin{aligned}
&= \frac{1}{2^{2N}} \sum_{\mu_1=0}^{3} \ldots \sum_{\mu_N=0}^{3} \sum_{\nu_1=0}^{3} \ldots \sum_{\nu_N=0}^{3} T_{\mu_1 \ldots \mu_N} T_{\nu_1 \ldots \nu_N} \mathrm{Tr} \left(\sigma_{\mu_1}^{(1)} \sigma_{\nu_1}^{(1)} \otimes \ldots \otimes \sigma_{\mu_N}^{(N)} \sigma_{\nu_N}^{(N)} \right) \\
&= \frac{1}{2^{2N}} \sum_{\mu_1=0}^{3} \ldots \sum_{\mu_N=0}^{3} \sum_{\nu_1=0}^{3} \ldots \sum_{\nu_N=0}^{3} T_{\mu_1 \ldots \mu_N} T_{\nu_1 \ldots \nu_N} \mathrm{Tr} \left(\sigma_{\mu_1}^{(1)} \sigma_{\nu_1}^{(1)} \right) \times \ldots \times \mathrm{Tr} \left(\sigma_{\mu_N}^{(N)} \sigma_{\nu_N}^{(N)} \right) \\
&= \frac{1}{2^{2N}} \sum_{\mu_1=0}^{3} \ldots \sum_{\mu_N=0}^{3} \sum_{\nu_1=0}^{3} \ldots \sum_{\nu_N=0}^{3} T_{\mu_1 \ldots \mu_N} T_{\nu_1 \ldots \nu_N} \left(2\delta_{\mu_1 \nu_1} \right) \ldots \left(2\delta_{\mu_N \nu_N} \right) \\
&= \frac{1}{2^{N}} \sum_{\mu_1=0}^{3} \ldots \sum_{\mu_N=0}^{3} T_{\mu_1 \ldots \mu_N}^2 . \qquad (1.17)
\end{aligned}
$$

From (1.16) and (1.17), one has for a pure state:

$$
\sum_{\mu_1=0}^{3} \ldots \sum_{\mu_N=0}^{3} T_{\mu_1 \ldots \mu_N}^2 = 2^N, \qquad (1.18)
$$

and for any general mixed state:

$$
\sum_{\mu_1=0}^{3} \ldots \sum_{\mu_N=0}^{3} T_{\mu_1 \ldots \mu_N}^2 \leq 2^N. \qquad (1.19)
$$

1.2 Objectives

The project aims to construct a complete set of monogamy relations, i.e. a set from which one is able to derive any other monogamy relations. On the way, we would like to find new monogamy relations that all physical states satisfy. In particular, we hope to find an alternative characterization of entanglement in pure states. Furthermore, we seek for the applications of the monogamy relations in the field of quantum communication.

1.3 Organization of the thesis

The thesis is organized as follows. After introducing the background of monogamy relation and quantum entanglement in Chapter 1, the complete set of monogamy relations is presented in Chapter 2. Chapter 3 then states new monogamy relations discovered along the project, especially a bound for bipartite correlations. Chapter 4 continues with a conjecture on multipartite correlations that would enable an alternative characterization of entanglement. Detailed proofs for the cases of up to three qubits as well as possible approaches for generalization are also discussed. Chapter 5 then considers Remote State Preparation and signification of the monogamy relations derived in the previous chapters. Finally, I conclude my thesis by summarizing the challenges of the study, the results of the project and the future works.

Chapter 2

Complete set of monogamy relations

2.1 Positive semi-definiteness of density operator

It is known that density operators are Hermitian and have all eigenvalues non-negative. Mathematically, one calls such matrices positive-semidefinite matrices. The converse is also true. Every positive-semidefinite matrix is the density operator of a physical state. Hence, checking the positive semi-definiteness of a matrix is sufficient to conclude the existence of a physical state. There are few algorithms to check for the positive-semidefiniteness of Hermitian matrices. In this section, the principal minors test will be introduced. Detailed proof can be found in [15],

Theorem 1. *A Hermitian matrix is positive-semidefinite if and only if all of its principal minors are non-negative.*

Minors of a matrix are the determinants of its submatrices. Principal minors are the determinants of submatrices obtained by deleting the same columns and rows. Determinant of the matrix itself is also considered a minor. For example,

$$\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}$$

are both minors of

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

but only the former one is a principal minor.

Let D_S ($S \subset \{1, 2, 3, ..., n\}, S \neq \emptyset$) be the determinant constructed from the same columns and rows specified by S. Condition of Theorem 1 can be rewritten as

$$D_S \geq 0 \ \ \forall S \subset \{1, 2, 3, ..., n\}, S \neq \emptyset. \tag{2.1}$$

These inequalities form a complete set in the sense that if a matrix satisfies all of them, it is a density operator representing a physical state. The job now is to write them in terms of correlation tensor and a complete set of monogamy relations is readily found.

2.2 Complete set of monogamy relations

In this section, we will present the complete set of monogamy relations for a two-qubit system. In that case, the density operator is generally a 4-by-4 matrix

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix}. \tag{2.2}$$

On the other hand, ρ can also be decomposed as in (1.14). From that, one may write the entries of ρ in terms of correlations

$$\rho_{11} = \frac{1}{4}(T_{00} + T_{03} + T_{30} + T_{33}), \tag{2.3}$$

$$\rho_{22} = \frac{1}{4}(T_{00} - T_{03} + T_{30} - T_{33}), \tag{2.4}$$

$$\rho_{33} = \frac{1}{4}(T_{00} + T_{03} - T_{30} - T_{33}), \tag{2.5}$$

$$\rho_{44} = \frac{1}{4}(T_{00} - T_{03} - T_{30} + T_{33}), \tag{2.6}$$

$$\rho_{12} = \rho_{21}^* = \frac{1}{4}\left[(T_{01} + T_{31}) + i(-T_{02} - T_{32})\right], \tag{2.7}$$

$$\rho_{13} = \rho_{31}^* = \frac{1}{4}\left[(T_{10} + T_{13}) + i(-T_{20} - T_{23})\right], \tag{2.8}$$

$$\rho_{14} = \rho_{41}^* = \frac{1}{4}\left[(T_{11} - T_{22}) + i(-T_{12} - T_{21})\right], \tag{2.9}$$

$$\rho_{23} = \rho_{32}^* = \frac{1}{4}\left[(T_{11} + T_{22}) + i(T_{12} - T_{21})\right], \tag{2.10}$$

$$\rho_{24} = \rho_{42}^* = \frac{1}{4}\left[(T_{10} - T_{13}) + i(-T_{20} + T_{23})\right], \tag{2.11}$$

$$\rho_{34} = \rho_{43}^* = \frac{1}{4}\left[(T_{01} - T_{31}) + i(-T_{02} + T_{32})\right], \tag{2.12}$$

where ρ_{21}^* is the complex conjugate of ρ_{12}. Hence every minor of ρ is a function of the correlation tensor T_{kl} and the set of inequalities (2.1) becomes a set of monogamy relations. The fact that the set is complete is a direct consequence of the completeness of (2.1). Here only inequalities involving 1-by-1 minors and 2-by-2 minors are listed down. The rest of the set are given in Appendix A as they are quite cumbersome. The 1-by-1 minors are actually the diagonal entries of ρ. Thus, corresponding monogamy relations are

$$\rho_{11} = \frac{1}{4}(T_{00} + T_{03} + T_{30} + T_{33}) \geq 0, \tag{2.13a}$$

$$\rho_{22} = \frac{1}{4}(T_{00} - T_{03} + T_{30} - T_{33}) \geq 0, \tag{2.13b}$$

$$\rho_{33} = \frac{1}{4}(T_{00} + T_{03} - T_{30} - T_{33}) \geq 0, \tag{2.13c}$$

$$\rho_{44} = \frac{1}{4}(T_{00} - T_{03} - T_{30} + T_{33}) \geq 0. \tag{2.13d}$$

Although the 2-by-2 minors are much more complicated, they are in fact more interesting as they involve second order of correlations. Direct calculation gives

$$D_{12} = \rho_{11}\rho_{22} - \rho_{12}\rho_{21} \geq 0,$$
$$\text{or } (T_{00} + T_{30})^2 \geq (T_{03} + T_{33})^2 + (T_{01} + T_{31})^2 + (T_{02} + T_{32})^2, \tag{2.14}$$

$$
\begin{aligned}
& D_{13} = \rho_{11}\rho_{33} - \rho_{13}\rho_{31} \geq 0, \\
\text{or } & (T_{00} + T_{03})^2 \geq (T_{30} + T_{33})^2 + (T_{10} + T_{13})^2 + (T_{20} + T_{23})^2, \qquad (2.15) \\
& D_{14} = \rho_{11}\rho_{44} - \rho_{14}\rho_{41} \geq 0, \\
\text{or } & (T_{00} + T_{33})^2 \geq (T_{03} + T_{30})^2 + (T_{11} - T_{22})^2 + (T_{12} + T_{21})^2, \qquad (2.16) \\
& D_{23} = \rho_{22}\rho_{33} - \rho_{23}\rho_{32} \geq 0, \\
\text{or } & (T_{00} - T_{33})^2 \geq (T_{03} - T_{33})^2 + (T_{11} + T_{22})^2 + (T_{12} - T_{21})^2, \qquad (2.17) \\
& D_{24} = \rho_{22}\rho_{44} - \rho_{42}\rho_{42} \geq 0, \\
\text{or } & (T_{00} - T_{03})^2 \geq (T_{30} - T_{33})^2 + (T_{10} - T_{13})^2 + (T_{20} - T_{32})^2, \qquad (2.18) \\
& D_{34} = \rho_{33}\rho_{44} - \rho_{34}\rho_{43} \geq 0, \\
\text{or } & (T_{00} - T_{30})^2 \geq (T_{03} - T_{33})^2 + (T_{01} - T_{31})^2 + (T_{02} - T_{32})^2. \qquad (2.19)
\end{aligned}
$$

One may also write (2.14)-(2.19) in more compact forms:

$$
\begin{aligned}
(T_{00} \pm T_{30})^2 &\geq (T_{03} \pm T_{33})^2 + (T_{01} \pm T_{31})^2 + (T_{02} \pm T_{32})^2, \qquad (2.20) \\
(T_{00} \pm T_{03})^2 &\geq (T_{30} \pm T_{33})^2 + (T_{10} \pm T_{13})^2 + (T_{20} \pm T_{23})^2, \qquad (2.21) \\
(T_{00} \pm T_{33})^2 &\geq (T_{03} \pm T_{30})^2 + (T_{11} \mp T_{22})^2 + (T_{12} \mp T_{21})^2. \qquad (2.22)
\end{aligned}
$$

Equation (2.14), for example, tells the trade off between $(T_{03} + T_{33})^2, (T_{01} + T_{31})^2$ and $(T_{02} + T_{32})^2$ as their sum is bounded by $(T_{00} + T_{30})^2$.

The completeness of the set implies that if one, in some ways, derives a monogamy relation, he must be able to do the same by combining inequalities from the set. The purity condition

$$\text{Tr}\left(\rho^2\right) \leq 1, \qquad (2.23)$$

is an example. One may obtain (2.23) just by summing up (2.14) to (2.19) to get

$$\text{Tr}\left(\rho^2\right) = \frac{1}{4}\left(\sum_{k,l=0}^{3} T_{kl}^2\right) \leq T_{00}^2 = 1 \qquad (2.24)$$

where the first equality is obtained by noticing that $\mathrm{Tr}\left(\rho^2\right)$ can be written in term of T_{kl} as done in equation (1.17). Furthermore, the state is pure, that is $\mathrm{Tr}\left(\rho^2\right) = 1$, if and only if equalities hold for equations (2.14) to (2.19), or all 2-by-2 minors are zeros, hence all but one of the eigenvalues of ρ must be zero [15] and the state must be pure.

It is worth mentioning that the monogamy relations derived here are stronger than the ones from correlation complementarity, which will be introduced in the next chapter. For example, since

$$\{\sigma_0 \otimes \sigma_3 + \sigma_3 \otimes \sigma_3, \sigma_0 \otimes \sigma_1 + \sigma_3 \otimes \sigma_1\} = 0, \tag{2.25}$$

$$\{\sigma_0 \otimes \sigma_3 + \sigma_3 \otimes \sigma_3, \sigma_0 \otimes \sigma_2 + \sigma_3 \otimes \sigma_2\} = 0, \tag{2.26}$$

$$\{\sigma_0 \otimes \sigma_1 + \sigma_3 \otimes \sigma_1, \sigma_0 \otimes \sigma_2 + \sigma_3 \otimes \sigma_2\} = 0, \tag{2.27}$$

where $\{\ \}$ is the anti-commutator, one has from the correlation complementarity:

$$(T_{03} + T_{33})^2 + (T_{01} + T_{31})^2 + (T_{02} + T_{32})^2 \leq 1. \tag{2.28}$$

However, from equation (2.14), one also has

$$(T_{03} + T_{33})^2 + (T_{01} + T_{31})^2 + (T_{02} + T_{32})^2 \leq (T_{00} + T_{30})^2 \leq 1, \tag{2.29}$$

which is clearly a better bound.

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Chapter 3

Correlation complementarity

3.1 Correlation complementarity

A bound on the sum of squared expectation values of anti-commuting operators in a quantum state is presented by Kurzynski *et.al.* [7, 16, 17].

Theorem 2. *Let*

$$S = \{A_j : j = 1, ..., k\}, \tag{3.1}$$

be a set of traceless and trace-orthogonal operators such that

$$A_j^2 = \mathbb{I} \quad (j = 1, ..., k), \tag{3.2}$$

and each operator anti-commutes with all other elements of the set

$$\{A_i, A_j\} = 0 \quad \forall i \neq j; \; i, j = 1, ..., k. \tag{3.3}$$

Let

$$\alpha_j = \langle A_j \rangle = Tr(\rho \, A_j) \, , \tag{3.4}$$

be the expectation value of A_j in ρ. Then

$$\sum_{A_j \in S} \alpha_j^2 \leq 1, \tag{3.5}$$

for any physical states ρ.

Note that if the operators A_j are the Pauli operators $\sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_N}$, α_j become the correlations defined in the Chapter 1. Thus, Theorem 2 tells the trade-offs between correlations in a physical state. Here we present a proof different from that in [7]. The proof makes uses of the positive-semidefiniteness of the density operator ρ.

Proof. Denote

$$\overrightarrow{\alpha} = (\alpha_1, \ldots, \alpha_k), \tag{3.6}$$

and

$$\overrightarrow{A} = (A_1, \ldots, A_k). \tag{3.7}$$

We will prove the theorem by contradiction. Assume that there exists a physical state represented by density operator ρ such that

$$\sum_{A_j \in S} \alpha_j^2 = |\overrightarrow{\alpha}|^2 > 1. \tag{3.8}$$

For convenience, define the operator

$$F = \sum_{j=1}^{k} \alpha_j A_j = \overrightarrow{\alpha}.\overrightarrow{A}. \tag{3.9}$$

Using conditions (3.2) and (3.3) one has

$$\begin{aligned} F^2 &= \left(\sum_{i=1}^{k} \alpha_i A_i\right)\left(\sum_{j=1}^{k} \alpha_j A_j\right) = \sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i \alpha_j A_i A_j \\ &= \sum_{j=1}^{k} \alpha_j^2 \mathbb{I} = |\overrightarrow{\alpha}|^2 \mathbb{I}. \end{aligned} \tag{3.10}$$

Therefore, F has only eigenvalues $\pm|\overrightarrow{\alpha}|$ and the two eigenvalues are of the same degeneracy since

$$\mathrm{Tr}(F) = \sum_{j=1}^{k} \alpha_j \mathrm{Tr}(A_j) = 0. \tag{3.11}$$

Consider unitary operator that diagonalizes F, i.e. $U^\dagger F U = |\overrightarrow{\alpha}|\,(\Pi_+ - \Pi_-)$, where $\Pi_\pm$ are the projectors onto subspaces of the degenerated eigenvalues $\pm|\overrightarrow{\alpha}|$:

$$\Pi_+ = \begin{pmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}, \quad \Pi_- = \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{pmatrix}. \tag{3.12}$$

Since $\Pi_+ + \Pi_- = \mathbb{I}$, one has

$$U\Pi_- U^\dagger = \frac{1}{2}\left(\mathbb{I} - \frac{1}{|\overrightarrow{\alpha}|}F\right). \tag{3.13}$$

Furthermore, let $S = \{A_j : j = 1, ..., k\}$ be extended to $S' = \{A_j : j = 1, ..., k, k+1, ..., n\}$ such that S', together with $\mathbb{I}$, forms a traceless and trace-orthogonal basis of the space of all states. Let ρ be decomposed in the basis as

$$\rho = \frac{1}{d}\left(\mathbb{I} + \sum_{j=1}^{n} \alpha_j A_j\right) = \frac{1}{d}\left(\mathbb{I} + F + \sum_{j=k+1}^{n} \alpha_j A_j\right), \tag{3.14}$$

where d is the dimension of the space. The probability to observe results associated with $U\Pi_- U^\dagger$ in the state ρ is

$$\begin{aligned} \mathrm{Tr}(\rho U\Pi_- U^\dagger) &= \mathrm{Tr}\left[\frac{1}{2d}\left(\mathbb{I} + F - \frac{1}{|\overrightarrow{\alpha}|}F - \frac{1}{|\overrightarrow{\alpha}|}F^2 + \sum_{j=k+1}^{n}\alpha_j A_j - \frac{1}{|\overrightarrow{\alpha}|}\sum_{j=k+1}^{n}\alpha_j F A_j\right)\right] \\ &= \frac{1}{2d}\left(1 - |\overrightarrow{\alpha}|\right). \end{aligned} \tag{3.15}$$

where (3.10) and the trace-orthogonality of the operators A_j have been applied. Since $|\overrightarrow{\alpha}| > 1$ by the assumption, the probability is negative, and it hence contradicts the fact that ρ is a physical state. The theorem follows. □

As shown in the proof, the correlation complementarity is a consequence of the fact that the probability of observing a physical quantity in a physical state is non-negative, or equivalently the density operator ρ is positive-semidefinite. Thus, it is tolerable that the correlation complementarity is weaker than inequalities from the complete set introducted in the last chapter. Still, this correlation complementarity theorem allows one to derive new monogamy relations, as long as one has a set of mutually anti-commuting operators. Such relations are listed in the next section. Some of them will be used frequently in the later chapters.

3.2 Some new monogamy relations

Theorem 3. *Every physical state of three or more particles satisfies:*

$$||T_A||^2 + ||T_{AB}||^2 + ||T_{AC}||^2 \leq 3, \tag{3.16}$$

where

$$\begin{aligned} ||T_A||^2 &\equiv \sum_{j=1}^{3} T^2_{j000...0}, \\ ||T_{AB}||^2 &\equiv \sum_{j,k=1}^{3} T^2_{jk00...0}, \\ ||T_{AC}||^2 &\equiv \sum_{j,l=1}^{3} T^2_{j0l0...0}. \end{aligned} \tag{3.17}$$

Proof. The proof comes immediately from the correlation complementarity theorem by realizing that correlations are, by definition, expectation values of Pauli operators. First, let the correlations be arranged into groups such that the operators that correspond to members of the same group anti-commute. One such rearrangement is

$$
\begin{array}{ccccccc}
 & & & & \text{Group 1} & \text{Group 2} & \text{Group 3} \\
T_{1000..0} & T_{1100..0} & T_{1010..0} & & T_{1000..0} & T_{2000..0} & T_{3000..0} \\
T_{2000..0} & T_{1200..0} & T_{1020..0} & & T_{2100..0} & T_{3100..0} & T_{1100..0} \\
T_{3000..0} & T_{1300..0} & T_{1030..0} & \Rightarrow & T_{2200..0} & T_{3200..0} & T_{1200..0} \\
 & T_{2100..0} & T_{2010..0} & & T_{2300..0} & T_{3300..0} & T_{1300..0} \\
 & T_{2200..0} & T_{2020..0} & & T_{3010..0} & T_{1010..0} & T_{2010..0} \\
 & T_{2300..0} & T_{2030..0} & & T_{3020..0} & T_{1020..0} & T_{2020..0} \\
 & T_{3100..0} & T_{3010..0} & & T_{3030..0} & T_{1030..0} & T_{2030..0} \\
 & T_{3200..0} & T_{3020..0} & & & & \\
 & T_{3300..0} & T_{3030..0} & & & &
\end{array}
$$

Since Pauli matrices mutually anti-commute with each other and commute with identity matrix, it can be verified that operators which correspond to members of the same group mutually anti-commute. Applying the correlation complementarity theorem to each group gives

$$T_{1000..0}^2 + T_{2100..0}^2 + T_{2200..0}^2 + T_{2300..0}^2 + T_{3010..0}^2 + T_{3020..0}^2 + T_{3030..0}^2 \leq 1, \tag{3.18}$$

$$T_{2000..0}^2 + T_{3100..0}^2 + T_{3200..0}^2 + T_{3300..0}^2 + T_{2010..0}^2 + T_{2020..0}^2 + T_{2030..0}^2 \leq 1, \tag{3.19}$$

$$T_{3000..0}^2 + T_{1100..0}^2 + T_{1200..0}^2 + T_{1300..0}^2 + T_{2010..0}^2 + T_{2020..0}^2 + T_{2030..0}^2 \leq 1. \tag{3.20}$$

Finally, the result (3.16) is obtained by adding up the three inequalities above. □

Corollary 1. *Every physical state of three or more particles satisfies:*

$$\frac{1}{2}(||T_A||^2 + ||T_B||^2 + ||T_C||^2) + ||T_{AB}||^2 + ||T_{AC}||^2 + ||T_{BC}||^2 \leq \frac{9}{2}. \tag{3.21}$$

Proof. From (3.16), by permuting A, B and C, we have the following inequalities

$$
\begin{aligned}
||T_A||^2 + ||T_{AB}||^2 + ||T_{AC}||^2 &\leq 3, \\
||T_B||^2 + ||T_{AB}||^2 + ||T_{BC}||^2 &\leq 3, \\
||T_C||^2 + ||T_{AC}||^2 + ||T_{BC}||^2 &\leq 3.
\end{aligned} \tag{3.22}
$$

By summing them up and dividing the sum by 2, we get the desired result. □

Theorem 4. *Every physical state of four or more particles satisfies:*

$$||T_{AB}||^2 + ||T_{AC}||^2 + ||T_{AD}||^2 \leq 3, \tag{3.23}$$

where $||T_{AB}||^2$*,* $||T_{AC}||^2$*,* $||T_{AD}||^2$ *are defined in the same manner as in (3.17).*

Proof. It is now the matter of rearranging the correlations into groups of mutually anti-commuting corresponding operators. One possible rearrangement is

$$\begin{array}{ccccccc} & & & & \text{Group 1} & \text{Group 2} & \text{Group 3} \\ T_{11000..0} & T_{10100..0} & T_{10010..0} & & T_{11000..0} & T_{10100..0} & T_{10010..0} \\ T_{12000..0} & T_{10200..0} & T_{10020..0} & & T_{12000..0} & T_{10200..0} & T_{10020..0} \\ T_{13000..0} & T_{10300..0} & T_{10030..0} & \Rightarrow & T_{13000..0} & T_{10300..0} & T_{10030..0} \\ T_{21000..0} & T_{20100..0} & T_{20010..0} & & T_{20100..0} & T_{20010..0} & T_{21000..0} \\ T_{22000..0} & T_{20200..0} & T_{20020..0} & & T_{20200..0} & T_{20020..0} & T_{22000..0} \\ T_{23000..0} & T_{20300..0} & T_{20030..0} & & T_{20300..0} & T_{20030..0} & T_{23000..0} \\ T_{31000..0} & T_{30100..0} & T_{30010..0} & & T_{30010..0} & T_{31000..0} & T_{30100..0} \\ T_{32000..0} & T_{30200..0} & T_{30020..0} & & T_{30020..0} & T_{32000..0} & T_{30200..0} \\ T_{33000..0} & T_{30300..0} & T_{30030..0} & & T_{30030..0} & T_{33000..0} & T_{30300..0} \end{array}$$

Now sum of square of elements in a group is not greater than 1 by the correlation complementarity theorem. Since there are three such groups, the result (3.23) follows.

□

Corollary 2. *Every physical state of four or more particles satisfies:*

$$||T_{AB}||^2 + ||T_{AC}||^2 + ||T_{AD}||^2 + ||T_{BC}||^2 + ||T_{BD}||^2 + ||T_{CD}||^2 \leq 6. \tag{3.24}$$

Proof. By permuting (3.23), we have the following inequalities

$$||T_{AB}||^2 + ||T_{AC}||^2 + ||T_{AD}||^2 \leq 3, \tag{3.25}$$

$$||T_{BC}||^2 + ||T_{BC}||^2 + ||T_{BD}||^2 \leq 3, \tag{3.26}$$

$$||T_{CA}||^2 + ||T_{CB}||^2 + ||T_{CD}||^2 \leq 3, \tag{3.27}$$

$$||T_{DA}||^2 + ||T_{DB}||^2 + ||T_{DC}||^2 \leq 3. \tag{3.28}$$

By summing them up and dividing the sum by 2 (since each correlation appears twice), the desired result (3.24) is obtained. □

Theorem 5. *Every physical state of four or more particles satisfies:*

$$6||T_A||^2 + ||T_{AB}||^2 + ||T_{ABC}||^2 + ||T_{ABD}||^2 \leq 9. \tag{3.29}$$

Proof. The proof is similar to the previous theorem. Let the correlations be grouped as follows

Grp 1	Grp 2	Grp 3	Grp 4	Grp 5	Grp 6	Grp 7	Grp 8	Grp 9
$T_{20000..0}$	$T_{20000..0}$	$T_{20000..0}$	$T_{30000..0}$	$T_{30000..0}$	$T_{30000..0}$	$T_{10000..0}$	$T_{10000..0}$	$T_{10000..0}$
$T_{30000..0}$	$T_{30000..0}$	$T_{30000..0}$	$T_{10000..0}$	$T_{10000..0}$	$T_{10000..0}$	$T_{20000..0}$	$T_{20000..0}$	$T_{20000..0}$
$T_{11000..0}$	$T_{12000..0}$	$T_{13000..0}$	$T_{21000..0}$	$T_{22000..0}$	$T_{23000..0}$	$T_{31000..0}$	$T_{32000..0}$	$T_{33000..0}$
$T_{12100..0}$	$T_{13100..0}$	$T_{11100..0}$	$T_{22100..0}$	$T_{23100..0}$	$T_{21100..0}$	$T_{32100..0}$	$T_{33100..0}$	$T_{31100..0}$
$T_{12200..0}$	$T_{13200..0}$	$T_{11200..0}$	$T_{22200..0}$	$T_{23200..0}$	$T_{21200..0}$	$T_{32200..0}$	$T_{33200..0}$	$T_{31200..0}$
$T_{12300..0}$	$T_{13300..0}$	$T_{11300..0}$	$T_{22300..0}$	$T_{23300..0}$	$T_{21300..0}$	$T_{32300..0}$	$T_{33300..0}$	$T_{31300..0}$
$T_{13010..0}$	$T_{11010..0}$	$T_{12010..0}$	$T_{23010..0}$	$T_{21010..0}$	$T_{22010..0}$	$T_{33010..0}$	$T_{31010..0}$	$T_{32010..0}$
$T_{13020..0}$	$T_{11020..0}$	$T_{12020..0}$	$T_{23020..0}$	$T_{21020..0}$	$T_{22020..0}$	$T_{33020..0}$	$T_{31020..0}$	$T_{32020..0}$
$T_{13030..0}$	$T_{11030..0}$	$T_{12030..0}$	$T_{23030..0}$	$T_{21030..0}$	$T_{22030..0}$	$T_{33030..0}$	$T_{31030..0}$	$T_{32030..0}$

Elements of the same group correspond to mutually anti-commuting operators. Hence their sum of squares is upper bounded by 1. Since there are 9 such groups, the result (3.29) follows. □

3.3 A bound on the sum of squared bipartite correlations

Bipartite correlations are the ones that involve only subsystems of two qubits and therefore have indices of the form $kl00...0$ and its permutations, where $k, l = 1, 2, 3$. The monogamy relations proved in the last section give an upper bound on the sum of squared bipartite correlations, as stated in the following theorem.

Theorem 6. *The sum of squared bipartite correlations of an n-qubit state* $(n \geq 4)$ *is not greater than* $\binom{n}{2}$. *Formally,*

$$\sum_{\pi} \sum_{k,l=1,2,3} T^2_{\pi(k,l,0,0,...,0)} \leq \binom{n}{2}, \tag{3.30}$$

where $\pi(k, l, 0, 0, ..., 0)$ *are permutations of the list.*

Proof. By Theorem 4, we have

$$||T_{AB}||^2 + ||T_{AC}||^2 + ||T_{AD}||^2 \leq 3. \tag{3.31}$$

Since there are n choices of A, $\binom{n-1}{3}$ choices of B, C, D, we have $n\binom{n-1}{3}$ such equations. Now each equation is bounded by 3, and bipartite correlations corresponding to each pair of qubits, $||T_{AB}||^2$ for example, appear $\binom{n-2}{2} + \binom{n-2}{2} = (n-2)(n-3)$ times. If we sum all such equations and divide it by $(n-2)(n-3)$, we get

$$\sum_{\pi} \sum_{k,l=1,2,3} T^2_{\pi(k,l,0,0,...,0)} \leq n\binom{n-1}{3} \cdot \frac{2}{(n-2)(n-3)} = \binom{n}{2}. \tag{3.32}$$

The inequality is saturated if the given state is fully separable. □

Remark In the case of four qubits, the inequality reduces to the one in Corollary 2. The proof follows exactly from the proof of the corollary.

Theorem 6 gives a stronger bound than a similar work by Markiewicz *et.al.* [18] which states that for any n-qubit states with $n \geq 3$, the following holds

$$\sum_{\pi} \sum_{k,l=1,2} T^2_{\pi(k,l,0,0,\ldots,0)} \leq \binom{n}{2}. \tag{3.33}$$

The difference between the two is that in the LHS of (3.33), the sum is taken over $k, l = 1, 2$ instead of $k, l = 1, 2, 3$ as in Theorem 6. It is remarkable that $\binom{n}{2}(3^2 - 2^2)$ non-negative terms are added to the LHS of (3.33) and yet the bound is still $\binom{n}{2}$. Without any doubts, Theorem 6 is a major improvement on the inequality presented in [18].

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Chapter 4

Characterization of entanglement

4.1 Conjecture on the sum of squared multipartite correlations

In the earlier period of the project, we were attracted to the problem of identification of entanglement in pure states. Although there is a solution, namely a pure state is entangled if and only if at least one of its subsystems is not in pure state, we would like to find an alternative identification which involves multipartite correlations, that are correlations between all the particles. So far we have obtained analytically such an identification for pure states of two and three particles and tested it extensively numerically for four particles.

Conjecture 1. *A pure state of N qubits is entangled if and only if:*

$$\sum_{j_1,\ldots,j_N=1}^{3} T^2_{j_1\ldots j_N} > 1. \tag{4.1}$$

The relation is non-trivial. As one shall see in the example given at the end of the next chapter, there exists a pure state, of which absolute values of multipartite correlations are all less than 1 and still the sum of squares of these correlations is greater than 1. Here the proofs for $N = 2$ and $N = 3$ using correlation complementarity and Schmidt decomposition are presented.

4.2 Schmidt decomposition proof

Theorem 7. (Schmidt decomposition [3]) *Suppose* $|\Psi\rangle$ *is a pure state of a composite system* AB*. Then there exist orthonormal states* $|\alpha_j\rangle$ *for system* A*, and orthonormal states* $|\beta_j\rangle$ *for system* B *such that*

$$|\Psi\rangle = \sum_j \lambda_j |\alpha_j\rangle \otimes |\beta_j\rangle, \tag{4.2}$$

where λ_j *are non-negative real numbers satisfying* $\sum_j \lambda_j^2 = 1$ *known as Schmidt coefficients*

Theorem 8. *Suppose* $|\Psi\rangle$ *is a pure state of a composite system* AB*. Let* $\rho_A = \mathrm{Tr}_B(\rho_{AB})$ *and* $\rho_B = \mathrm{Tr}_A(\rho_{AB})$*. Then*

$$\mathrm{Tr}(\rho_A^2) = \mathrm{Tr}(\rho_B^2) \leq 1, \tag{4.3}$$

Proof. Let

$$|\Psi\rangle = \sum_j \lambda_j |\alpha_j\rangle \otimes |\beta_j\rangle, \tag{4.4}$$

be the Schmidt decomposition of $|\Psi\rangle$. The density operator of the composite system AB is

$$\rho_{AB} = |\Psi\rangle\langle\Psi| = \sum_j \sum_k \lambda_j \lambda_k |\alpha_j\rangle\langle\alpha_k| \otimes |\beta_j\rangle\langle\beta_k|. \tag{4.5}$$

Density operators of subsystems are

$$\begin{aligned}\rho_A &= \mathrm{Tr}_B(\rho_{AB}) = \mathrm{Tr}_B\left(\sum_j \sum_k \lambda_j \lambda_k |\alpha_j\rangle\langle\alpha_k| \otimes |\beta_j\rangle\langle\beta_k|\right) \\ &= \sum_j \sum_k \lambda_j \lambda_k |\alpha_j\rangle\langle\alpha_k| \delta_{j,k} = \sum_j \lambda_j^2 |\alpha_j\rangle\langle\alpha_j|,\end{aligned} \tag{4.6}$$

$$\begin{aligned}\rho_B &= \mathrm{Tr}_A(\rho_{AB}) = \mathrm{Tr}_A\left(\sum_j \sum_k \lambda_j \lambda_k |\alpha_j\rangle\langle\alpha_k| \otimes |\beta_j\rangle\langle\beta_k|\right) \\ &= \sum_j \sum_k \lambda_j \lambda_k \delta_{j,k} |\beta_j\rangle\langle\beta_k| = \sum_j \lambda_j^2 |\beta_j\rangle\langle\beta_j|.\end{aligned} \tag{4.7}$$

Therefore,

$$\mathrm{Tr}\left(\rho_A^2\right) = \mathrm{Tr}(\sum_j \lambda_j^4 \left|\alpha_j\right\rangle \left\langle\alpha_j\right|) = \sum_j \lambda_j^4 = \mathrm{Tr}(\sum_j \lambda_j^4 \left|\beta_j\right\rangle \left\langle\beta_j\right|) = \mathrm{Tr}\left(\rho_B^2\right). \tag{4.8}$$

Furthermore,

$$\mathrm{Tr}\left(\rho_A^2\right) = \mathrm{Tr}\left(\rho_B^2\right) = \sum_j \lambda_j^4 \leq \left(\sum_j \lambda_j^2\right)^2 \leq 1. \tag{4.9}$$

The equality holds if and only if all but one of the Schmidt coefficients λ_j are zeros, i.e. the state is a product state. □

Theorem 9. (Conjecture 1 for $N = 2$**)**
A pure state of two qubits is entangled if and only if:

$$\sum_{k,l=1}^{3} T_{kl}^2 > 1. \tag{4.10}$$

Proof. Since the state $|\Psi\rangle$ is pure, one has from (1.18):

$$4 = \sum_{\mu,\nu=0}^{3} T_{\mu\nu}^2 = T_{00}^2 + \sum_{k=1}^{3} T_{k0}^2 + \sum_{l=1}^{3} T_{0l}^2 + \sum_{k,l=1}^{3} T_{kl}^2. \tag{4.11}$$

Since $T_{00} = 1$ as always,

$$\sum_{k=1}^{3} T_{k0}^2 + \sum_{l=1}^{3} T_{0l}^2 + \sum_{k,l=1}^{3} T_{kl}^2 = 3. \tag{4.12}$$

Now let ρ_{AB} be decomposed in the basis of Pauli matrices

$$\rho_{AB} = \frac{1}{4}\left(\mathbb{I} + \sum_{k=1}^{3} T_{k0}\sigma_k^{(A)} \otimes \sigma_0^B + \sum_{l=1}^{3} T_{0l}\sigma_0^{(A)} \otimes \sigma_l^{(B)} + \sum_{k,l=1}^{3} T_{kl}\sigma_k^{(A)} \otimes \sigma_l^{(B)}\right). \tag{4.13}$$

Density operators of the subsystems are

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \frac{1}{2}\left(\mathbb{I} + \sum_{k=1}^{3} T_{k0}\sigma_k^{(A)}\right), \tag{4.14}$$

$$\rho_B = \mathrm{Tr}_A(\rho_{AB}) = \frac{1}{2}\left(\mathbb{I} + \sum_{l=1}^{3} T_{0l}\sigma_k^{(B)}\right). \tag{4.15}$$

From Theorem 8, one has

$$\mathrm{Tr}(\rho_A^2) = \frac{1}{2}\left(1 + \sum_{k=1}^{3} T_{k0}^2\right) \leq 1 \tag{4.16}$$

$$\iff \quad \sum_{k=1}^{3} T_{k0}^2 \leq 1. \tag{4.17}$$

Similarly,

$$\sum_{l=1}^{3} T_{0l}^2 \leq 1. \tag{4.18}$$

Substituting the two above inequalities into equation (4.12) gives

$$\sum_{k,l=1}^{3} T_{kl}^2 \geq 1. \tag{4.19}$$

The equality holds if and only if $\mathrm{Tr}\left(\rho_A^2\right) = \mathrm{Tr}\left(\rho_B^2\right) = 1$, if and only if all but one of the Schmidt coefficients λ_j are zero, which means $|\Psi\rangle = \lambda_j |\alpha_j\rangle \otimes |\beta_j\rangle$ for some j is non-entangled. The theorem follows. □

Theorem 10. (Conjecture 1 for $N = 3$**)** *A pure state of three qubits is entangled if and only if:*

$$\sum_{k,l,m=1}^{3} T_{klm}^2 > 1. \tag{4.20}$$

Proof. As done in the proof of Theorem 9, the purity of $|\Psi\rangle$ gives

$$\begin{aligned} 8 &= \sum_{\mu,\nu,\eta=0}^{3} T_{\mu\nu\eta}^2 \\ &= T_{000}^2 + \sum_{k=1}^{3} T_{k00}^2 + \sum_{l=1}^{3} T_{0l0}^2 + \sum_{m=1}^{3} T_{00m}^2 + \sum_{k,l=1}^{3} T_{kl0}^2 \\ &+ \sum_{k,m=1}^{3} T_{k0m}^2 + \sum_{l,m=1}^{3} T_{0lm}^2 + \sum_{k,l,m=1}^{3} T_{klm}^2. \end{aligned} \tag{4.21}$$

Since $T_{000} = 1$ as always,

$$\begin{aligned}\sum_{k=1}^{3} T_{k00}^2 + \sum_{l=1}^{3} T_{0l0}^2 + \sum_{m=1}^{3} T_{00m}^2 + \sum_{k,l=1}^{3} T_{kl0}^2 \\ + \sum_{k,m=1}^{3} +T_{k0m}^2 + \sum_{l,m=1}^{3} T_{0lm}^2 + \sum_{k,l,m=1}^{3} T_{klm}^2 = 7.\end{aligned} \tag{4.22}$$

Now one may consider the system ABC as a combination of subsystem A and subsystem BC. Define

$$\rho_A = \mathrm{Tr}_{BC}(\rho_{ABC}) \quad = \frac{1}{2}\left(\mathbb{I} + \sum_{k=1}^{3} T_{k00}\sigma_k^{(A)}\right), \tag{4.23}$$

$$\begin{aligned}\rho_{BC} = \mathrm{Tr}_A(\rho_{ABC}) \quad &= \frac{1}{4}(\mathbb{I} + \sum_{l=1}^{3} T_{0l0}\sigma_l^{(B)} \otimes \sigma_0^{(C)} \\ &+ \sum_{m=1}^{3} T_{00m}\sigma_0^{(B)} \otimes \sigma_m^{(C)} + \sum_{l,m=1}^{3} T_{0lm}\sigma_l^{(B)} \otimes \sigma_m^{(C)}).\end{aligned} \tag{4.24}$$

to be the density operators of the respective subsystem. Applying Schmidt decomposition to the cut $A|BC$ gives

$$\mathrm{Tr}\left(\rho_A^2\right) = \mathrm{Tr}\left(\rho_{BC}^2\right) \tag{4.25}$$

$$\iff \quad \frac{1}{2}\left(1 + \sum_{k=1}^{3} T_{k00}^2\right) = \frac{1}{4}\left(1 + \sum_{l=1}^{3} T_{0l0}^2 + \sum_{m=1}^{3} T_{00m}^2 + \sum_{l,m=1}^{3} T_{0lm}^2\right) \tag{4.26}$$

$$\iff \quad \sum_{l,m=1}^{3} T_{0lm}^2 = 1 + 2\sum_{k=1}^{3} T_{k00}^2 - \sum_{l=1}^{3} T_{0l0}^2 - \sum_{m=1}^{3} T_{00m}^2. \tag{4.27}$$

In the same manner, Schmidt decompositions for the cut $AB|C$ and $AC|B$ give

$$\sum_{k,l=1}^{3} T_{kl0}^2 = 1 + 2\sum_{m=1}^{3} T_{00m}^2 - \sum_{k=1}^{3} T_{k00}^2 - \sum_{l=1}^{3} T_{0l0}^2, \tag{4.28}$$

$$\sum_{k,m=1}^{3} T_{k0m}^2 = 1 + 2\sum_{l=1}^{3} T_{0l0}^2 - \sum_{m=1}^{3} T_{00m}^2 - \sum_{k=1}^{3} T_{k00}^2. \tag{4.29}$$

Summing up (4.27)-(4.29) gives

$$\sum_{l,m=1}^{3} T_{0lm}^2 + \sum_{k,l=1}^{3} T_{kl0}^2 + \sum_{k,m=1}^{3} T_{k0m}^2 = 3. \tag{4.30}$$

Substituting back to equation (4.22), one has

$$\sum_{k=1}^{3} T_{k00}^2 + \sum_{l=1}^{3} T_{0l0}^2 + \sum_{m=1}^{3} T_{00m}^2 + \sum_{k,l,m=1}^{3} T_{klm}^2 = 4. \tag{4.31}$$

Once again, Schmidt decomposition gives

$$\mathrm{Tr}(\rho_A^2) = \frac{1}{2}\left(1 + \sum_{k=1}^{3} T_{k00}^2\right) \leq 1 \iff \sum_{k=1}^{3} T_{k00}^2 \leq 1, \tag{4.32}$$

$$\mathrm{Tr}(\rho_B^2) = \frac{1}{2}\left(1 + \sum_{l=1}^{3} T_{0l0}^2\right) \leq 1 \iff \sum_{l=1}^{3} T_{0l0}^2 \leq 1, \tag{4.33}$$

$$\mathrm{Tr}(\rho_C^2) = \frac{1}{2}\left(1 + \sum_{m=1}^{3} T_{00m}^2\right) \leq 1 \iff \sum_{m=1}^{3} T_{00m}^2 \leq 1. \tag{4.34}$$

Therefore,

$$\sum_{k,l,m=1}^{3} T_{klm}^2 \geq 1. \tag{4.35}$$

The equality holds if and only if $\sum_{k=1}^{3} T_{k00}^2 = \sum_{l=1}^{3} T_{0l0}^2 = \sum_{m=1}^{3} T_{00m}^2$, if and only if ρ_A, ρ_B and ρ_C are all pure states, if and only if $|\Psi\rangle = |\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle$ is not entangled. $\square$

4.3 Correlation Complementarity proof

The proof for the case $N = 3$ using correlation complementarity is a direct consequence of Corollary 1. Since $|\Psi\rangle$ is pure,

$$\begin{aligned}\sum_{k=1}^{3} T_{k00}^2 + \sum_{l=1}^{3} T_{0l0}^2 + \sum_{m=1}^{3} T_{00m}^2 + \sum_{k,l=1}^{3} T_{kl0}^2 \\ + \sum_{k,m=1}^{3} + T_{k0m}^2 + \sum_{l,m=1}^{3} T_{0lm}^2 + \sum_{k,l,m=1}^{3} T_{klm}^2 = 7.\end{aligned} \tag{4.36}$$

By Corollary 1,

$$\frac{1}{2}\left(\sum_{k=1}^{3} T_{k00}^2 + \sum_{l=1}^{3} T_{0l0}^2 + \sum_{m=1}^{3} T_{00m}^2\right) + \sum_{k,l=1}^{3} T_{kl0}^2 + \sum_{k,m=1}^{3} T_{k0m}^2 + \sum_{l,m=1}^{3} T_{0lm}^2 \leq \frac{9}{2}. \tag{4.37}$$

At the same time, Schmidt decomposition gives

$$\mathrm{Tr}(\rho_A^2) = \frac{1}{2}\left(1 + \sum_{k=1}^{3} T_{k00}^2\right) \leq 1 \iff \sum_{k=1}^{3} T_{k00}^2 \leq 1, \tag{4.38}$$

$$\mathrm{Tr}(\rho_B^2) = \frac{1}{2}\left(1 + \sum_{l=1}^{3} T_{0l0}^2\right) \leq 1 \iff \sum_{l=1}^{3} T_{0l0}^2 \leq 1, \tag{4.39}$$

$$\mathrm{Tr}(\rho_C^2) = \frac{1}{2}\left(1 + \sum_{m=1}^{3} T_{00m}^2\right) \leq 1 \iff \sum_{m=1}^{3} T_{00m}^2 \leq 1. \tag{4.40}$$

Substitute (4.38)-(4.40) and (4.37) back in (4.36), one has

$$\sum_{k,l,m=1}^{3} T_{klm}^2 \geq 1. \tag{4.41}$$

The equality holds if and only if $|\Psi\rangle$ is a product state as before.

Correlation complementarity seems to be able to generalize to multipartite case by finding appropriate sets of mutually anti-commuting operators. However, the task is not as simple as one may think. Even with the help of computer, we have not yet found these sets that enable a complete proof even for four particles.

4.4 PDE approach

With the help of Mathematica, we have found an interesting property of the sum

$$S = \sum_{j_1,...,j_N=1}^{3} T^2_{j_1 \cdots j_N}. \tag{4.42}$$

that may enable a proof of the conjecture for any N.
First, let $|\alpha_j\rangle$ be a basis of the associated Hilbert space of pure states. Any pure state can be decomposed in the basis

$$|\Psi\rangle = \sum_{j}^{d} c_j |\alpha_j\rangle , \tag{4.43}$$

where $c_j = a_j + ib_j$ $(a_j, b_j \in \mathbb{R})$ are complex coefficients and d is the dimension of the space. The density operator $\rho = |\Psi\rangle \langle\Psi|$ is then a function of a_j and b_j,

$$\rho = \rho(a_1, ..., a_d, b_1, ..., b_d), \tag{4.44}$$

and so are the correlations

$$T_{j_1 j_2 \cdots j_N} = \mathrm{Tr}\left(\rho\sigma_{j_1} \otimes \cdots \otimes \sigma_{j_N}\right) = T_{j_1 j_2 \cdots j_N}(a_1, ..., a_d, b_1, ..., b_d). \tag{4.45}$$

As a consequence, $S = S(a_1, ..., a_d, b_1, ..., b_d)$ is also a function of a_j and b_j. Using Mathematica, we have shown for $N = 1, 2, 3, 4, 5$ that

$$\sum_{j}^{d} \left(\frac{\partial^2 S}{\partial a_j^2} + \frac{\partial^2 S}{\partial b_j^2} \right) = C(N). \tag{4.46}$$

where $C(N)$ is a positive constant number that depends only on N. The fact that S satisfies the above partial differential equation may be used to derive some properties of S. For example, Hopf maximal principle states that solutions of a harmonic partial differential equation cannot attain maximum or minimum inside an open set. However, for them to work, we need better understandings of differential geometry as well as partial differential equations.

4.5 Future work

As discussed, both proofs using Schmidt decomposition and correlation complementarity have reached dead ends. It is not known whether a proof from any of the two methods exists for four particles. Even if it does, there is no guarantee that it can be generalized. However, numerical sampling suggests the conjecture holds for $N = 4$ and it seems to be the case also for larger N. A possible new approach is to use Hopf maximal principle. We have proved that the sum in the conjecture is a solution of a harmonic-like partial differential equation.The principle states that a solution of such a differential equation does not achieve maximum or minimum in an open set, in this case the set of entangled states. Any maximum or minimum, if exist, must be on the boundary, which is the separable states. Though there are still ambiguous mathematical details that need to be verify, if true, the conjecture will hold for any pure states.

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Chapter 5

Remote State Preparation

In this chapter, we shall give a meaning to the conjecture proposed in the last chapter. It turns out to be related to the fidelity of Remote State Preparation. Furthermore, using monogamy relations derived in Chapter 3, we shall prove a bound on the total fidelity of the protocol.

Remote State Preparation (RSP) refers to mechanisms in which one wishes to prepare a quantum state at remote site using local operations, classical communication and a quantum entangled state [19]. In this chapter, a protocol to prepare states at remotes site using entangled systems will be introduced. The RSP protocol is similar to the one of quantum teleportation [20], except that in RSP the person knows the state which is meant to be prepared at the remote site while in quantum teleportation he does not. We shall then calculate fidelities of the results and apply the monogamy relations derived in the previous chapters to obtain a bound on the total fidelity.

5.1 Protocol of Remote State Preparation

The RSP protocol is as follows. Let there be a 2-qubit entangled state AB that is shared between Alice and Bob so that Alice has the qubit A and Bob has the qubit B. Now Alice would like to prepare a designated state at Bob's site by following this protocol. First, Alice obtains a third qubit S, so that she now has two qubits SA in her hand. This is the difference between RSP and quantum teleportation mentioned earlier. In RSP, Alice is allowed to choose the state of the third qubit S such that the final result at Bob's site is the closest to the designated state. Meanwhile in quantum teleportation, the third qubit given to Alice is in an unknown state. Next, Alice measures her two qubits in Bell basis:

$$\left|\Phi^+\right\rangle = \frac{1}{\sqrt(2)}\left(|0\rangle_S \otimes |0\rangle_A + |1\rangle_S \otimes |1\rangle_A\right), \tag{5.1}$$

$$\left|\Phi^-\right\rangle = \frac{1}{\sqrt(2)}\left(|0\rangle_S \otimes |0\rangle_A - |1\rangle_S \otimes |1\rangle_A\right), \tag{5.2}$$

$$\left|\Psi^+\right\rangle = \frac{1}{\sqrt(2)}\left(|0\rangle_S \otimes |1\rangle_A + |1\rangle_S \otimes |0\rangle_A\right), \tag{5.3}$$

$$\left|\Psi^-\right\rangle = \frac{1}{\sqrt(2)}\left(|0\rangle_S \otimes |1\rangle_A - |1\rangle_S \otimes |0\rangle_A\right). \tag{5.4}$$

The outcome will be one of the four states listed above. Alice then sends the result to Bob. Depending on the result, Bob applies an operator according to the rule listed here, on his qubit to obtain the designated state:

Alice's result		Operator Bob uses
$\|\Phi^+\rangle$	$\rightarrow$	σ_0
$\|\Phi^-\rangle$	$\rightarrow$	σ_3
$\|\Psi^+\rangle$	$\rightarrow$	σ_1
$\|\Psi^-\rangle$	$\rightarrow$	σ_2

If the system AB is prepared in a state $|\Phi^+\rangle$, the final qubit at Bob's site B' will be in the exact same state as the one Alice chooses at the beginning, S. Therefore, the protocol allows Alice to prepare arbitrary state at Bob's with absolute certainty. However, when AB is only partially entangled, the fidelity of the preparation protocol is typically less than 1 and depends on how AB are entangled as well as the state Alice would like to prepare.

5.2 Fidelity of Remote State Preparation

Let us now calculate the fidelity of the protocol above. First, we shall find the resulting state at Bob's site. Let the initial system shared between Alice and Bob, AB, be decomposed as in equation (1.14):

$$\rho_{AB} = \frac{1}{4}\sum_{\mu,\nu=0}^{3} T_{\mu\nu}\sigma_{\mu}^{(A)} \otimes \sigma_{\nu}^{(B)}. \tag{5.5}$$

Let the state Alice chooses, S, be represented by a Bloch vector $\overrightarrow{S} = (S_1, S_2, S_3)$ such that

$$\rho_S = \frac{1}{2}\left(\mathbb{I}^{(\mathbb{S})} + S_1.\sigma_1^{(S)} + S_2.\sigma_2^{(S)} + S_3.\sigma_3^{(S)}\right) = \frac{1}{2}\sum_{\lambda=0}^{3} S_\lambda.\sigma_\lambda^{(S)}, \tag{5.6}$$

where $S_0 = 1$ and $\sigma_0^{(S)} = \mathbb{I}^{(\mathbb{S})}$ is the identity.
The joined state of the three qubits is then

$$\rho_{SAB} = \rho_S \otimes \rho_{AB} = \frac{1}{8}\sum_{\mu,\nu,\lambda=0,1,2,3} S_\lambda T_{\mu\nu}\sigma_\lambda^{(S)} \otimes \sigma_\mu^{(A)} \otimes \sigma_\nu^{(B)}. \tag{5.7}$$

Now Alice measures her qubits SA in the Bell basis and obtains one of the four Bell states. In general, let the Bell states be represented by

$$\rho_{Bell} = \frac{1}{4}\sum_{\lambda',\mu'=0}^{3} E_{\lambda'\mu'}\sigma_{\lambda'}^{(S)} \otimes \sigma_{\mu'}^{(A)}, \tag{5.8}$$

where $E_{\lambda'\mu'}$ is the correlation tensor of ρ_{Bell}. Each of the Bell states has a respective correlation tensor $E_{\lambda'\mu'}$. Then Bob's qubit will collapse to B' represented by

$$\begin{aligned}
\rho_{B'} &= \mathrm{Tr}_{SA}\left(\rho_{SAB}\rho_{Bell}\right) \\
&= \mathrm{Tr}_{SA}\left(\frac{1}{32}\sum_{\mu,\nu,\lambda,\mu',\nu'=0}^{3} E_{\lambda'\mu'}S_\lambda T_{\mu\nu}\sigma_\lambda^{(S)}\sigma_{\lambda'}^{(S)} \otimes \sigma_\mu^{(A)}\sigma_{\mu'}^{(A)} \otimes \sigma_\nu^{(B)}\right) \\
&= \frac{1}{8}\sum_{\mu,\nu,\lambda=0}^{3} E_{\lambda'\mu'}S_\lambda T_{\mu\nu}\delta_{\lambda\lambda'}\delta_{\mu\mu'}\sigma_\nu^{(B)}
\end{aligned}$$

$$= \frac{1}{8} \sum_{\mu,\nu,\lambda=0}^{3} E_{\lambda\mu} S_\lambda T_{\mu\nu} \sigma_\nu^{(B)}, \tag{5.9}$$

where δ is the Kronecker delta. From (1.13) and (5.1)-(5.4), one may obtain the correlation tensors E corresponding to each of the four Bell states:

$$E(|\Phi^+\rangle) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.10}$$

$$E(|\Phi^-\rangle) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.11}$$

$$E(|\Psi^+\rangle) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{5.12}$$

$$E(|\Psi^-\rangle) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{5.13}$$

Substituting (5.10)-(5.13) into (5.9), a little simplification and renormalization gives the state $\rho_{B'}$ of Bob's qubit corresponding to the four possible outcomes of Alice's measurement:

$$\rho_{B'}(|\Psi^-\rangle) = \frac{1}{2}(\mathbb{I} + \frac{\overrightarrow{b} - (S_1, S_2, S_3)T_0}{1 - (S_1, S_2, S_3).\overrightarrow{a}}.\overrightarrow{\sigma}), \tag{5.14}$$

$$\rho_{B'}(|\Psi^+\rangle) = \frac{1}{2}(\mathbb{I} + \frac{\overrightarrow{b} + (S_1, S_2, -S_3)T_0}{1 + (S_1, S_2, -S_3).\overrightarrow{a}}.\overrightarrow{\sigma}), \tag{5.15}$$

$$\rho_{B'}(\left|\Phi^{+}\right\rangle) = \frac{1}{2}(\mathbb{I} + \frac{\overrightarrow{b} + (S_1, -S_2, S_3)T_0}{1 + (S_1, -S_2, S_3).\overrightarrow{a}}.\overrightarrow{\sigma}), \tag{5.16}$$

$$\rho_{B'}(\left|\Phi^{-}\right\rangle) = \frac{1}{2}(\mathbb{I} + \frac{\overrightarrow{b} + (-S_1, S_2, S_3)T_0}{1 + (-S_1, S_2, S_3).\overrightarrow{a}}.\overrightarrow{\sigma}), \tag{5.17}$$

where

$$\overrightarrow{a} = (T_{10}, T_{20}, T_{30}), \tag{5.18}$$

$$\overrightarrow{b} = (T_{01}, T_{02}, T_{03}), \tag{5.19}$$

$$\overrightarrow{\sigma} = (\sigma_1, \sigma_2, \sigma_3), \tag{5.20}$$

are the local Bloch vectors of A and B respectively, $\overrightarrow{\sigma}$ is the vector of Pauli operators, and

$$T_0 = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}. \tag{5.21}$$

Then Bob decides which operator to apply according to the rule listed above. The resulting states ρ_r are

$$\rho_r(\left|\Phi^{+}\right\rangle) = \frac{1}{2}(\mathbb{I} + \frac{(-b_x, b_y, -b_z) - \overrightarrow{S}T_0}{1 + (S_1, -S_2, S_3).\overrightarrow{a}}.\overrightarrow{\sigma}), \tag{5.22}$$

$$\rho_r(\left|\Phi^{-}\right\rangle) = \frac{1}{2}(\mathbb{I} + \frac{(b_x, -b_y, -b_z) - \overrightarrow{S}T_0}{1 + (-S_1, S_2, S_3).\overrightarrow{a}}.\overrightarrow{\sigma}), \tag{5.23}$$

$$\rho_r(\left|\Psi^{+}\right\rangle) = \frac{1}{2}(\mathbb{I} + \frac{(-b_x, -b_y, b_z) - \overrightarrow{S}T_0}{1 + (S_1, S_2, -S_3).\overrightarrow{a}}.\overrightarrow{\sigma}), \tag{5.24}$$

$$\rho_r(\left|\Psi^{-}\right\rangle) = \frac{1}{2}(\mathbb{I} + \frac{(b_x, b_y, b_z) - \overrightarrow{S}T_0}{1 - (S_1, S_2, S_3).\overrightarrow{a}}.\overrightarrow{\sigma}). \tag{5.25}$$

Their respective Bloch vectors are

$$\overrightarrow{r}_{|\Phi^{+}\rangle} = \frac{(-b_x, b_y, -b_z) - \overrightarrow{S}T_0}{1 + (S_1, -S_2, S_3).\overrightarrow{b}}, \tag{5.26}$$

$$\overrightarrow{r}_{|\Phi^{-}\rangle} = \frac{(b_x, -b_y, -b_z) - \overrightarrow{S}T_0}{1 + (-S_1, S_2, S_3).\overrightarrow{b}}, \tag{5.27}$$

$$\overrightarrow{r}_{|\Psi^+\rangle} = \frac{(-b_x, -b_y, b_z) - \overrightarrow{S} T_0}{1 + (S_1, S_2, -S_3).\, \overrightarrow{b}}, \tag{5.28}$$

$$\overrightarrow{r}_{|\Psi^-\rangle} = \frac{(b_x, b_y, b_z) - \overrightarrow{S} T_0}{1 - (S_1, S_2, S_3).\, \overrightarrow{b}}. \tag{5.29}$$

Now, let the probability that the Alice's outcome is $|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle$ and $|\Psi^-\rangle$ be $P_{|\Phi^+\rangle}$, $P_{|\Phi^-\rangle}$, $P_{|\Psi^+\rangle}$ and $P_{|\Psi^-\rangle}$ respectively. One has

$$P_{|\Phi^+\rangle} = \text{Tr}\left(\left|\Phi^+\right\rangle\left\langle\Phi^+\right|\ \rho_{SA}\right) = \frac{1}{4}[1 - (S_1, -S_2, S_3).\overrightarrow{a}], \tag{5.30}$$

$$P_{|\Phi^-\rangle} = \text{Tr}\left(\left|\Phi^-\right\rangle\left\langle\Phi^-\right|\ \rho_{SA}\right) = \frac{1}{4}[1 - (-S_1, S_2, S_3).\overrightarrow{a}], \tag{5.31}$$

$$P_{|\Psi^+\rangle} = \text{Tr}\left(\left|\Psi^+\right\rangle\left\langle\Psi^+\right|\ \rho_{SA}\right) = \frac{1}{4}[1 - (S_1, S_2, -S_3).\overrightarrow{a}], \tag{5.32}$$

$$P_{|\Psi^-\rangle} = \text{Tr}\left(\left|\Psi^-\right\rangle\left\langle\Psi^-\right|\ \rho_{SA}\right) = \frac{1}{4}[1 - (S_1, S_2, S_3).\overrightarrow{a}]. \tag{5.33}$$

The average resulting state at Bob's site is then

$$\overline{\overrightarrow{r}} = P_{|\Psi-\rangle}\overrightarrow{r}_{|\Psi^-\rangle} + P_{|\Psi+\rangle}\overrightarrow{r}_{|\Psi^+\rangle} + P_{|\Phi-\rangle}\overrightarrow{r}_{|\Phi^-\rangle} + P_{|\Phi^+\rangle}\overrightarrow{r}_{|\Phi^+\rangle} \tag{5.34}$$

$$= -\overrightarrow{S} T_0. \tag{5.35}$$

We define the fidelity as the square of inner product between the Bloch vector of the average resulting state $\overline{\overrightarrow{r}}$ and $\overrightarrow{m} = (m_1, m_2, m_3)$, the Bloch vector of the state Alice actually wishes to prepare at Bob's site:

$$F = \left(\overline{\overrightarrow{r}}.\overrightarrow{m}\right)^2 = \left(\overrightarrow{S} T_0.\overrightarrow{m}\right)^2 = \left(\overrightarrow{S}.\overrightarrow{m}T_0^T\right)^2. \tag{5.36}$$

Since Alice has the freedom to choose the qubit A, and hence the corresponding Bloch vector $\overrightarrow{S}$, she will do it in a way such that the fidelity is maximized:

$$\begin{aligned} F_{max} &= \max_{\overrightarrow{S}}\left\{\left(\overrightarrow{S}.\overrightarrow{m}T_0^T\right)^2\right\} \\ &= |\overrightarrow{m}T_0^T|^2 \qquad \left(\text{when } \overrightarrow{S} \parallel \overrightarrow{m}T_0^T\right) \end{aligned}$$

$$= (T_{11}m_1 + T_{12}m_2 + T_{13}m_3)^2 + (T_{21}m_1 + T_{22}m_2 + T_{23}m_3)^2$$
$$+ (T_{31}m_1 + T_{32}m_2 + T_{33}m_3)^2. \tag{5.37}$$

At the end of this chapter, an example is given showing that sometimes it is optimal for Alice to choose the state $\overrightarrow{S}$ of the third qubit different from $\overrightarrow{m}$. Since only the characteristic of the system AB is of interest, let the fidelity be averaged over every possible state that Alice wishes to prepare at Bob's site, $\overrightarrow{m}$

$$\overline{F_{max}} = \frac{1}{4\pi} \iint_{\Omega} F_{max} d\overrightarrow{m} = \frac{1}{3} \sum_{\mu,\nu=1}^{3} T_{\mu,\nu}^2 = \frac{1}{3} ||T_{AB}||^2, \tag{5.38}$$

where the integral is taken over the whole Bloch sphere. The average maximum fidelity is hence related to the bipartite correlations between A and B. The protocol can be extended to the situation where Alice would like to prepare simultaneously states at multiple remote sites, namely Bob, Charlie and David. In this case, they share an entangled four-qubit system $ABCD$ such that A belongs to Alice, B belongs to Bob, C belongs to Charlie and finally D belongs to David. The sum of fidelities is then

$$\overline{F_{tot}} = \overline{F_{AB}} + \overline{F_{AC}} + \overline{F_{AD}} \tag{5.39}$$
$$= \frac{1}{3}||T_{AB}||^2 + \frac{1}{3}||T_{AC}||^2 + \frac{1}{3}||T_{AD}||^2 \tag{5.40}$$
$$= \frac{1}{3}\left(||T_{AB}||^2 + ||T_{AC}||^2 + ||T_{AD}||^2\right) \tag{5.41}$$
$$\leq \frac{1}{3}\, 3 = 1. \tag{5.42}$$

Here Theorem 4 is applied in the last line. Thus this bound tells us that the final states at Bob, Charlie and Dave could not be prepared perfectly ($\overline{F}_{AB} = \overline{F}_{AC} = \overline{F}_{AD} = 1$) at the same time. If Alice is in favor of preparing the state at Bob's site closer to the designated state, she will have to do it at the cost of the fidelities of the states at Charlie's and David's.

As promised, we shall now give an example showing that it is optimal for Alice to choose the state $\overrightarrow{S}$ of the third qubit different from the one she wishes to prepare at Bob's site, $\overrightarrow{m}$.

Example: Suppose Alice wishes to prepare the state $\overrightarrow{m} = \frac{1}{\sqrt{3}}(1,1,1)$ and the state shared between Alice and Bob is

$$|\Psi\rangle = \frac{1}{2}|00\rangle + \frac{\sqrt{3}}{2}|11\rangle\,, \tag{5.43}$$

which has the correlation matrix

$$T_0 = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}. \tag{5.44}$$

If Alice chooses $\overrightarrow{s} = \overrightarrow{m} = \frac{1}{\sqrt{3}}(1,1,1)$ as her third state, the fidelity is

$$F = |\overrightarrow{m} T_0.\overrightarrow{m}|^2 = \frac{1}{4}. \tag{5.45}$$

Meanwhile, if she chooses $\overrightarrow{s}' = \frac{1}{\sqrt{3}}(1,-1,1)$,

$$F' = |\overrightarrow{s'} T_0.\overrightarrow{m}|^2 = \frac{3}{4} > \frac{1}{4}. \tag{5.46}$$

Therefore it is beneficial for Alice to choose the state of the third qubit different from the one she wishes to prepare at Bob's site.

Chapter 6

Conclusion

In this project, we have studied the monogamy relations from different perspectives. First, we have established a complete set of monogamy relations in the sense that any Hermitian matrix which satisfies all the monogamy relations in the set must be a density operator of a physical state. The set has been derived from the positive-semidefiniteness of the density operator. We have also given an example, the purity condition, to demonstrate the fact that every monogamy relation is a combination of monogamy relations from the complete set, and we showed, by comparing with correlation complementarity, how the complete relations are more restrictive than other relations derived up to date.

In the next chapter, we have started from correlation complementarity, which is a monogamy relation itself, to derive several monogamy relations concerning the trade-offs between correlations. From one of those, we have proved the existence of an upper bound on the sum of squared bipartite correlations in a physical system of any number, greater than four, of qubits. The bound is tight and turns out to be much stronger than a similar bound obtained by Markiewicz *et.al.* [18].

Next, we have proposed a conjecture that would enable an alternative identification of entanglement. The conjecture only involves correlations between all of the parties. Proofs for systems of two and three qubits, using Schmidt decomposition and correlation complementarity, have been demonstrated but none of them seems to be able to generalize to systems of any numbers of qubits. However, numerical simulation suggests the conjecture does hold for larger number of qubits. One promising approach to prove the conjecture is to utilize the Hopf maximal principle. However, many hypotheses which are essential for the principle to apply need to be verified. Finally, we gave a meaning to the quantities entering the conjecture by proposing a protocol to remotely prepare quantum states and the fidelity, which expresses its performance, has been shown to depend on the correlations of a quantum state used in the protocol. Furthermore, using the bound we have just derived, we showed that in the case of remote state preparation at three different sites, the sum of fidelities of the prepared states at all sites is not greater than one, and hence disallows the three states to be prepared perfectly at the same time.

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Appendix A

Complete set of monogamy relations

This appendix lists all monogamy relations which are consequences of the positivity of 3-by-3 minors and 4-by-4 minors.

$\boxed{D_{123}}$

$$\begin{aligned}
&\left[(T_{00}+T_{30})^2-(T_{01}+T_{31})^2-(T_{02}+T_{32})^2-(T_{03}+T_{33})^2\right](T_{00}+T_{03}-T_{30}-T_{33})\\
&-\left[(T_{11}+T_{22})^2+(T_{12}-T_{21})^2\right](T_{00}+T_{03}+T_{30}+T_{33})\\
&-\left[(T_{10}+T_{13})^2+(T_{20}-T_{23})^2\right](T_{00}-T_{03}+T_{30}-T_{33})\\
&+(T_{10}+T_{13}+iT_{20}+iT_{23})(T_{01}+T_{31}-iT_{02}-iT_{32})(T_{11}+T_{22}+iT_{12}-iT_{21})\\
&+(T_{10}+T_{13}-iT_{20}-iT_{23})(T_{01}+T_{31}+iT_{02}+iT_{32})(T_{11}+T_{22}-iT_{12}+iT_{21})\\
&\geq 0.
\end{aligned} \tag{A.1}$$

$\boxed{D_{124}}$

$$\begin{aligned}
&\left[(T_{00}+T_{30})^2-(T_{01}+T_{31})^2-(T_{02}+T_{32})^2-(T_{03}+T_{33})^2\right](T_{00}-T_{03}-T_{30}+T_{33})\\
&-\left[(T_{11}-T_{22})^2+(T_{12}+T_{21})^2\right](T_{00}-T_{03}+T_{30}-T_{33})\\
&-\left[(T_{10}-T_{13})^2+(T_{20}-T_{23})^2\right](T_{00}+T_{03}+T_{30}+T_{33})\\
&+(T_{10}-T_{13}-iT_{20}+iT_{23})(T_{01}+T_{31}-iT_{02}-iT_{32})(T_{11}-T_{22}+iT_{12}+iT_{21})\\
&+(T_{10}-T_{13}+iT_{20}-iT_{23})(T_{01}+T_{31}+iT_{02}+iT_{32})(T_{11}-T_{22}-iT_{12}-iT_{21})\\
&\geq 0.
\end{aligned} \tag{A.2}$$

$\boxed{D_{134}}$

$$\begin{aligned}
&\left[(T_{00}+T_{03})^2-(T_{10}+T_{13})^2-(T_{20}+T_{23})^2-(T_{30}+T_{33})^2\right](T_{00}-T_{03}-T_{30}+T_{33})\\
&-\left[(T_{11}-T_{22})^2+(T_{12}+T_{21})^2\right](T_{00}+T_{03}-T_{30}-T_{33})\\
&-\left[(T_{01}-T_{31})^2+(T_{02}-T_{32})^2\right](T_{00}+T_{03}+T_{30}+T_{33})\\
&+(T_{10}+T_{13}-iT_{20}-iT_{23})(T_{01}-T_{31}-iT_{02}-iT_{32})(T_{11}-T_{22}+iT_{12}+iT_{21})\\
&+(T_{10}+T_{13}+iT_{20}+iT_{23})(T_{01}-T_{31}+iT_{02}+iT_{32})(T_{11}-T_{22}-iT_{12}-iT_{21})\\
&\geq 0.
\end{aligned} \tag{A.3}$$

$\boxed{D_{234}}$

$$\begin{aligned}
&\left[(T_{00}-T_{33})^2-(T_{03}-T_{30})^2-(T_{12}-T_{21})^2-(T_{11}+T_{22})^2\right](T_{00}-T_{03}+T_{30}+T_{33})\\
&-\left[(T_{10}-T_{13})^2+(T_{20}-T_{23})^2\right](T_{00}+T_{03}-T_{30}-T_{33})\\
&-\left[(T_{01}-T_{31})^2+(T_{02}-T_{32})^2\right](T_{00}-T_{03}+T_{30}-T_{33})\\
&+(T_{10}-T_{13}+iT_{20}-iT_{23})(T_{01}-T_{31}-iT_{02}+iT_{32})(T_{11}+T_{22}+iT_{12}-iT_{21})\\
&+(T_{10}-T_{13}-iT_{20}+iT_{23})(T_{01}-T_{31}+iT_{02}-iT_{32})(T_{11}+T_{22}-iT_{12}+iT_{21})\\
&\geq 0.
\end{aligned} \tag{A.4}$$

$\boxed{D_{1234}}$

$$
\begin{aligned}
&\left[(T_{11}+iT_{21})^2-(T_{22}-iT_{12})^2\right]\left[(T_{20}+iT_{10})^2+(T_{11}-iT_{21})^2+(T_{12}-iT_{22})^2+(T_{13}-iT_{23})^2\right]\\
&+\left[(T_{00}-T_{30})^2-(T_{03}-T_{33})^2\right]\left[(T_{00}+T_{30})^2-(T_{01}+T_{31})^2-(T_{02}+T_{32})^2+(T_{03}+T_{33})^2\right]\\
&-\left[(T_{01}-T_{31})^2+(T_{02}-T_{32})^2\right]\left[(T_{00}+T_{30})^2-(T_{01}+T_{31})^2-(T_{02}+T_{32})^2+(T_{03}+T_{33})^2\right]\\
&+\left[(T_{10}+iT_{20})^2-(T_{13}-iT_{23})^2\right]\left[(T_{10}-iT_{20})^2+(T_{21}+iT_{11})^2+(T_{22}+iT_{12})^2+(T_{23}+iT_{13})^2\right]\\
&-\left[(T_{11}+iT_{12})^2-(T_{22}-iT_{21})^2\right]\left[(T_{01}-iT_{02})^2-(T_{31}-iT_{32})^2\right]\\
&-\left[(T_{11}-iT_{12})^2-(T_{22}+iT_{21})^2\right]\left[(T_{01}+iT_{02})^2-(T_{31}+iT_{32})^2\right]\\
&-\left[(T_{11}-T_{22})^2+(T_{12}+T_{21})^2\right]\left[(T_{00}-T_{33})^2-(T_{03}-T_{30})^2\right]\\
&-\left[(T_{11}+T_{22})^2+(T_{12}-T_{21})^2\right]\left[(T_{00}+T_{33})^2-(T_{03}+T_{30})^2\right]\\
&-\left[(T_{10}+T_{13})^2+(T_{20}+T_{23})^2\right]\left[(T_{00}-T_{03})^2-(T_{30}-T_{33})^2\right]\\
&-\left[(T_{10}-T_{13})^2+(T_{20}-T_{23})^2\right]\left[(T_{00}+T_{03})^2-(T_{30}+T_{33})^2\right]\\
&-\left[(T_{10}+iT_{23})^2-(T_{13}+iT_{20})^2\right]\left[(T_{01}-iT_{32})^2-(T_{31}-iT_{02})^2\right]\\
&-\left[(T_{10}-iT_{23})^2-(T_{13}-iT_{20})^2\right]\left[(T_{01}+iT_{32})^2-(T_{31}+iT_{02})^2\right]\\
&+(T_{11}-T_{22}+iT_{12}+iT_{21})(T_{01}-T_{31}-iT_{02}+iT_{32})(T_{10}+T_{13}-iT_{20}-iT_{23})\times\\
&\times(T_{00}-T_{03}+T_{30}-T_{33})+(T_{11}-T_{22}+iT_{12}+iT_{21})(T_{01}+T_{31}-iT_{02}-iT_{32})\times\\
&\times(T_{10}-T_{13}-iT_{20}+iT_{23})(T_{00}+T_{03}-T_{30}-T_{33})+(T_{11}+T_{22}+iT_{12}iT_{21})\times\\
&\times(T_{01}+T_{31}-iT_{02}-iT_{32})(T_{10}+T_{13}+iT_{20}+iT_{23})(T_{00}-T_{03}-T_{30}+T_{33})\\
&+(T_{11}+T_{22}-iT_{12}+iT_{21})(T_{01}+T_{31}+iT_{02}+iT_{32})(T_{10}+T_{13}-iT_{20}-iT_{23})\times\\
&\times(T_{00}-T_{03}T_{30}+T_{33})+(T_{11}-T_{22}-iT_{12}-iT_{21})(T_{01}-T_{31}+iT_{02}-iT_{32})\times\\
&\times(T_{10}+T_{13}+iT_{20}+iT_{23})(T_{00}-T_{03}+T_{30}-T_{33})+(T_{11}+T_{22}-iT_{12}+iT_{21})\times\\
&\times(T_{01}-T_{31}+iT_{02}-iT_{32})(T_{10}-T_{13}-iT_{20}+iT_{23})(T_{00}+T_{03}+T_{30}+T_{33})\\
&+(T_{11}+T_{22}+iT_{12}-iT_{21})(T_{01}-T_{31}-iT_{02}+iT_{32})(T_{10}-T_{13}+iT_{20}-iT_{23})\times\\
&\times(T_{00}+T_{03}+T_{30}+T_{33})+(T_{11}-T_{22}-iT_{12}-iT_{21})(T_{01}+T_{31}+iT_{02}+iT_{32})\times\\
&\times(T_{10}-T_{13}+iT_{20}-iT_{23})(T_{00}+T_{03}+T_{30}+T_{33})\geq 0.
\end{aligned}
\tag{A.5}
$$